2 Chaotic dynamics and itineraries

What is chaos?

There are several definitions in the literature. We’ll take perhaps the most accepted one.

The dynamical system must have:

- infinitely many different periodic orbits of different prime periods.
- a dense orbit
- sensitive dependence on initial conditions (SDIC)

Recall: a subset $X$ of $S$ is dense if given any open set $U$ of $S$ (no matter how small) there is at least one point of $X$ contained in $U$. Intuitively, no matter how closely you look, $X$ is spread all over $S$. (E.g., rational numbers are dense in $\mathbb{R}$.)

Definition of sensitive dependence on initial conditions: $f : S \to S$ exhibits SDIC at $x_0$ if there is a $d > 0$ such that no matter how close to $x_0$ you take another initial point $y_0$, eventually $x_n$ and $y_n$ are at least a distance $d$ apart:

$$\exists d > 0 \text{ such that } \forall \varepsilon > 0 \text{ there is } y_0 \in N_{\varepsilon}(x_0) \text{ and } n > 0 \text{ for which } y_n \notin N_d(x_n).$$

And $f$ exhibits SDIC if it exhibits SDIC at every point in $S$. In other words, there is a critical distance $d > 0$ such that however close two points begin, they are eventually iterated a distance at least $d$ apart. (In principle $d$ can depend on $x$, but in practice that’s not necessary.)

Example A rather trivial example of SDIC is provided by $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$. Take any value you like for $d$, e.g. $d = 20$. If $|x_0 - y_0| = \varepsilon > 0$ then $|x_1 - y_1| = 2\varepsilon$ and $|x_2 - y_2| = 2^2 \varepsilon$ etc. Now choose $n$ large enough that $2^n \varepsilon > 20$ and you’re done: $|x_n - y_n| > 20$.

But this is not a very “chaotic” example: the dynamics is very simple and not what one would think of as chaos at all! Which is why the definition needs the other ingredients too (or others like them).

In general, it is hard to prove that a given dynamical system is chaotic. And there are several important systems for which mathematicians are trying to prove this and have not (yet) succeeded. But there are some where it is not so difficult: one is the full logistic map $f_4$ (i.e., $\mu = 4$) and the tent map $T$. We will prove it for these maps, using the technique of itineraries.

2.1 Itineraries and Transition Graphs

Idea/example: the Tent map, $T : [0, 1] \to [0, 1]$ with

$$T(x) = \begin{cases} 2x & \text{if } x \in L \\ 2 - 2x & \text{if } x \in R \end{cases}$$

where $L = [0, 1/2]$ and $R = [1/2, 1]$, so $S = L \cup R$.

Note that if $x_0 \in [0, 1/4]$ then $x_0 \in L$ and $x_1 \in L$. We say $x_0 \in LL$. That is,

$$LL = \{x_0 \in [0, 1] \mid x_0 \in L, x_1 \in L\}.$$

Similarly,

$$LLR = \{x_0 \in [0, 1] \mid x_0 \in L, x_1 \in L, x_2 \in R\}.$$

And so on. (Check that $LLR = \left[ \frac{1}{8}, \frac{1}{4} \right]$.)
The general setting is as follows:

Let \( f : S \to S \) be a dynamical system (don’t need continuous at first), and suppose
\[
S = S_1 \cup S_2 \cup \ldots \cup S_N
\]
We will be assuming that the \( S_j \) are compact intervals.

The itinerary of an initial point \( x_0 \in S \) is the sequence
\[
A_0, A_1, A_2, \ldots, A_n, \ldots
\]
where \( x_j \in A_j \) (each \( A_j \) being one of \( S_1, \ldots, S_N \)). So an itinerary is just a sequence of symbols.

If we write a finite list of these symbols (with no commas),
\[
A_0 A_1 A_2 \ldots A_n
\]
this means the subset of \( S \) consisting of those points whose itinerary begins with this sequence (without any further assumptions of course, this set may be empty).

With \( f : S \to S \), and \( S = S_1 \cup S_2 \cup \ldots \cup S_N \) (as before). The transition graph is then the graph whose vertices are \( S_1, \ldots, S_N \) and we draw an arrow from \( S_i \) to \( S_j \) whenever
\[
S_j \subset f(S_i).
\]
This is called the covering rule. So \( \blacktriangleleft \) means that each point in \( B \) has a point in \( A \) which maps to it; in other words given any \( b \in B \) there is a solution of the equation \( f(a) = b \) with \( a \in A \).

\[\textbf{Example}\] (1) The map with graph on the right has transition graph below
\[\begin{array}{c}
\text{A} \quad \text{B} \\
\end{array}\]
The list \( ABC \) means the set of points \( x_0 \in A \) with \( x_1 \in B \) and \( x_2 \in C \). The list \( ACB \) corresponds to the empty set!

(2) BUT, this slightly modified version of the map (with graph on right) now has this transition graph (because \( B \) is no longer contained in \( f(A) \), even though some points of \( B \) are in \( f(A) \)).

\[\textbf{Definition}\] An allowed path in a transition graph is a path through the vertices following the arrows. In the first graph above, for example, \( ABC \) is an allowed path, as is \( ABCABBBB \); however \( ABAC \) is not (and corresponds to the empty set).

In the example (1) above, \( ABCA \) and \( ABBBCA \) are allowed, but of course \( ACB \) is not.

\[\textbf{Example}\] [Tent map] The transition graph of the tent map is
\[\text{L} \leftrightarrow \text{R} \leftrightarrow \text{L} \leftrightarrow \text{R} \]
In this case, any sequence of \( L \)s and \( R \)s is allowed.
Given a point, it is easy to find its itinerary: just apply the map a few times and watch where the orbit goes. What about the converse? Do we know whether an allowed path always corresponds to a non-empty subset of \( S \)? The answer is “yes”:

**Lemma 2.1** Let \( A_0A_1A_2 \ldots A_k \) be an allowed path in the transition graph of \( f \). Then the set \( A_0A_1A_2 \ldots A_k \subset S \) is non-empty. In other words, there is a point \( x_0 \in A_0 \) whose itinerary follows the sequence: \( x_1 \in A_1 \), then \( x_2 \in A_2 \), etc., until \( x_k \in A_k \).

**Proof** Start at the end: let \( a \in A_k \). Since \( A_{k−1}A_k \) is part of an allowed path, there exists \( b \in A_{k−1} \) with \( a = f(b) \). Similarly (since \( A_{k−2}A_{k−1} \) is part of an allowed path), there is \( c \in A_{k−2} \) such that \( b = f(c) \). And so on (by induction). Finally ending up with \( x \in A_0 \). Call this \( x_0 \). So \( x_0 \in A_0 \).

And \( x_1 = f(x_0) \in A_1 \) and so on, with \( b = x_{k−1} = f^{k−1}(x_0) \in A_{k−1} \) and \( a = x_k = f^k(x_0) \in A_k \). □

Note that \( A_0A_1A_2 \ldots A_k \) is not necessarily an interval. However it contains an interval, say \( I \), such that \( f^k(I) \) covers \( A_k \) (by intermediate value theorem).

### 2.2 Transition graphs and periodic orbits

**Theorem 2.2 (Second fixed point theorem)** Let \( f : S \to S \) be a continuous dynamical system, where \( S \subset \mathbb{R} \). Suppose there is a compact interval \( J \subset S \) such that \( f^k(J) \supset J \). Then \( f \) has a fixed point in \( J \).

**Proof** Compact means closed and bounded in this setting, so we write \( J = [x_a, x_b] \). Because \( f \) is continuous and \( J \) is compact, \( f \) is bounded on \( J \) and actually attains its bounds at certain points in \( J \). We can, therefore, define \( x_{\text{min}}, x_{\text{max}} \in J \) as points where \( f \) attains its minimum and maximum values respectively. By hypothesis, \( f^k(J) \supset J \) and so it follows that \( f(x_{\text{min}}) \leq x_a \) and \( f(x_{\text{max}}) \geq x_b \). Now define the function \( g(x) = f(x) − x \). This is continuous because \( f \) is continuous. We therefore have the following inequalities:

\[
\begin{align*}
g(x_{\text{min}}) &= f(x_{\text{min}}) − x_{\text{min}} \leq x_a − x_{\text{min}} \leq 0 \\
g(x_{\text{max}}) &= f(x_{\text{max}}) − x_{\text{max}} \geq x_b − x_{\text{max}} \geq 0
\end{align*}
\]

and thus, by the intermediate value theorem, there exists an \( x \), which lies between \( x_{\text{min}} \) and \( x_{\text{max}} \), such that \( g(x) = 0 \). This \( x \) is a fixed point of \( f \) since \( g(x) = 0 \Rightarrow f(x) = x \). □

It is worth comparing this proof with that of the earlier fixed point theorem. Although they look similar, they are different.

**Corollary 2.3** Let \( A_0A_1A_2 \ldots A_{k−1}A_0 \) be an allowed path in the transition graph of a 1-dimensional dynamical system. The there is a periodic point \( x_0 \in A_0 \) of period \( k \) whose itinerary follows the allowed path: \( A_0A_1A_2 \ldots A_{k−1} \).

**Example** In the tent map, where \( S = L \cup R \), the allowed path \( LRL \) shows the existence of period-2 orbits with itinerary \( \overline{LR} \). More generally the allowed path \( LR^{n−1}L \) shows the existence of period-\( n \) orbits. Furthermore, since the transition graphs of the logistic map (with \( \mu = 4 \)) is the same, the same paths prove the existence of similar periodic orbits.

**Proof** As has already been noted, \( A_0A_1A_2 \ldots A_{k−1}A_0 \) is not necessarily an interval itself, but may consist of many distinct intervals at least one of which, say \( J \), is such that \( f^k(J) \supset A_k \) (which equals \( A_0 \) in this case). Now \( J \subset A_0 \) and since \( f^k(J) \supset A_0 \) we have that \( f^k(J) \supset J \). It now
follows from the theorem above that \( f^k \) has a fixed point \( x_0 \in J \). Since \( x_0 \in J \) so \( x_0 \in A_0 \), \( x_1 \in A_1 \), \( x_2 \in A_2 \) etc. And a fixed point of \( f^k \) is a point of period \( k \). □

**Corollary** The logistic and tent maps have periodic orbits of every period.

**Uniqueness** In some cases one can make statements saying that a there is only one point with a particular itinerary. This needs a hypothesis involving the following definition:

We will say that a transition graph is **contracting** if, given any allowed infinite path \( A_1 A_2 \ldots \), the length of the interval tends to zero:

\[
\text{length}(A_1 A_2 \ldots A_k) \xrightarrow{k \to \infty} 0
\]

(if in \( \mathbb{R}^2, \mathbb{R}^3 \) or \( \mathbb{R}^n \), can replace ‘length’ by ‘area’, ‘volume’, or ‘measure’).

**Proposition 2.4** If a dynamical system has a contracting transition graph, and \( A_1 A_2 \ldots A_k \ldots \) is an allowed infinite path, then there is only one point in \( S \) with that itinerary.

So, in particular, if \( A_0 A_1 A_2 \ldots A_{k-1} A_0 \) is an allowed path, and the transition graph is contracting, there is only one periodic point following that itinerary.

**Exercise** Show that every genuine periodic orbit (ie not a fixed point) of a dynamical system with a contracting transition graph must visit more than one of the nodes of the transition graph. ♣

**Proposition 2.5** The transition graphs of the tent map and the full logistic map (\( \mu = 4 \)) are both contracting.

**Proof** For the tent map, can prove by induction that

\[
\text{length}(A_1 A_2 \ldots A_k) = \frac{1}{2^k}
\]

For the logistic map instead,

\[
\text{length}(A_1 A_2 \ldots A_k) \leq \frac{1}{2^{k-1}}.
\]

This will follow later, when we show that the tent map and the logistic map are conjugate. □

**Remark**: Paradoxically, it is the expanding nature of these maps that make the transition graphs contracting!

### 2.3 Transition graphs and dense orbits

We show that the Tent map and the Logistic map both have a dense orbit.

These two maps have the same transition graph \( \xleftrightarrow{\text{L}} \xleftrightarrow{\text{R}} \).

Let \( x_0 \) be the point with itinerary \( \text{LRLLRLRLRLRRLLLRLRRRLLRRRRRR} \ldots \) That is:

\[
\xleftrightarrow{\text{L}} \text{R} \xleftrightarrow{\text{L}} \text{LR} \xleftrightarrow{\text{R}} \text{L} \xleftrightarrow{\text{R}} \text{RR} \xleftrightarrow{\text{L}} \text{LL} \xleftrightarrow{\text{RLL}} \ldots
\]

This itinerary contains *every* finite sequence of L’s and R’s, so the orbit will visit every open interval in \([0, 1]\). That is, the orbit of \( x_0 \) is dense in \([0, 1]\).
2.4 Transition graphs and SDIC

Recall definition of SDIC above.

For logistic and tent maps, take $d = 1/4$.

Suppose we are given any $x_0 \in [0, 1]$, and let its itinerary be $A_0 A_1 A_2 \ldots$, where each $A_j$ is either $L$ or $R$.

Now let $y_0$ have itinerary $B_0 B_1 B_2 \ldots$ where $B_j = A_j$ for $j < k$ and $B_j \neq A_j$ for $j \geq k$. ($k$ will be chosen later, depending on $\epsilon$).

Then $x_k \in A_k A_{k+1}$ while $y_k \in B_k B_{k+1}$. Look at the possibilities (table on right).

For example, if $x_k \in LR$ then $y_k \in RL$, and these two intervals are separated by the interval RR.

In the tent map $RR = [1/2, 3/4]$ and $LR = [1/4, 1/2]$ both of which have length $1/4$. In the logistic map, RR and LR are both longer than $1/4$. In either case therefore,

$$|x_k - y_k| > 1/4 = d.$$

Now to $k$: since the transition graphs of the two maps in question are contracting, we can choose $k$ large enough so that $|x_0 - y_0| < \epsilon$ (for tent map, need $2^k > \frac{1}{\epsilon}$ and for logistic it's $2^k > \frac{2}{1 - 2\epsilon}$).

We have therefore proved that both the tent and logistic maps exhibit SDIC, and putting this together with the previous results we conclude that they are chaotic.

2.5 Period 3 implies chaos

This is a famous theorem of Li and Yorke (from 1975), and is where the word chaos was first used (in a mathematical context).

The definition they used was not the same as the one we have taken, but only asks for SDIC at infinitely many points and infinitely many periodic orbits of different periods.

**Theorem 2.6** Let $f : S \to S$ be continuous, with $S \subset \mathbb{R}$, and suppose $f$ has a period three orbit. Then $f$ is chaotic (in the sense of Li and Yorke).

We just prove the periodic orbit part of this theorem.

**Proof** Uses itinerary method.

Let $(a, b, c)$ be the period-3 orbit, and suppose $a < b < c$. There are two possibilities: either $f(a) = b$ or $f(a) = c$. In the former, $f(b) = c, f(c) = a$, while in the latter $f(c) = b, f(b) = a$. We will discuss the former case, leaving the latter as an exercise.

Write $L = [a, b]$ and $R = [b, c]$. The transition graph is then

1). This map has the following periodic orbits:

- $RRR \ldots = \bar{R}$ — fixed point;
- $RLRL \ldots = \bar{RL}$ — period 2
RRLRRL... = \mathbb{R}^2L — period 3

... \\
\mathbb{R}^nL — prime period n + 1.

So it has an orbit of any given period—inﬁnitely many in all. (There are also others: eg both \text{LRL}^4 and \text{LRLR}^2 have period 5.)

\begin{remark}
The hypotheses are not suﬃcient to guarantee the existence of a dense orbit, for example the period 3 orbit may be attracting so that any orbit that gets within \( \varepsilon \) of that orbit stays there and can’t visit every open set in the state space.
\end{remark}

2.6 Sarkovskii’s theorem

We saw in the last section how the existence of a period 3 orbit implies the existence of orbits of every other period.

Is the same true if there’s an orbit of period 5? The answer is ‘no’, but it does imply the existence of periodic orbits of all periods except 3. Similarly a period 7 orbit implies the existence of periodic orbits of all periods except 3 and 5. And a period-4 orbit implies just the existence of a period-2 orbit and a ﬁxed point. This was all discovered by Sarkovskii, and he found a general rule governing which periods force existence of which others:

Consider the following ordering of the natural numbers (called the Sarkovskii ordering):

\[
3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright 11 \triangleright \cdots
\]
\[
\triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright 2 \cdot 9 \triangleright 2 \cdot 11 \triangleright \cdots
\]
\[
\triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright 2^2 \cdot 9 \triangleright 2^2 \cdot 11 \triangleright \cdots
\]
\[
\triangleright 2^3 \cdot 3 \triangleright 2^3 \cdot 5 \triangleright 2^3 \cdot 7 \triangleright 2^3 \cdot 9 \triangleright 2^3 \cdot 11 \triangleright \cdots
\]
\[
\vdots
\]
\[
\vdots
\]
\[
\cdots \triangleright 2^n \triangleright \cdots \triangleright 16 \triangleright 8 \triangleright 4 \triangleright 2 \triangleright 1.
\]

(Can read \( n \triangleright k \) as \( n \) comes before \( k \).)

Of course, every natural number is a product of a power of 2 and an odd number, so this contains all natural numbers. That this is an ordering means that it’s transitive: so for example \( 3 \triangleright 6 \) and \( 6 \triangleright 4 \).

\begin{theorem}[Sarkovskii’s theorem]
Let \( f : I \rightarrow I \) be a continuous dynamical system on an interval, with an orbit of period \( n \). If \( n \triangleright k \) then \( f \) also has an orbit of period \( k \).
\end{theorem}

There are a number of proofs of this theorem — one using the method of itineraries, similar to our proof for period 3.
2.7 Conjugacy of dynamical systems

Example Consider the logistic map $f(x) = 4x(1-x)$, with $x \in S = [0, 1]$, but suppose we want to use the variable $u = 2x - 1$ instead. Then $u \in [-1, 1]$ (as $u = -1$ when $x = 0$ and $u = 1$ when $x = 1$). We want to write $u_{n+1} = g(u_n)$ from $x_{n+1} = f(x_n)$:

\[
\begin{align*}
    u_{n+1} &= 2x_{n+1} - 1 \quad \text{(definition of $u$)} \\
    &= 8x_n(1-x_n) - 1 \quad (x_{n+1} = f(x_n)) \\
    &= 8\left(\frac{1}{2}u_{n+1}\right)\left(1 - \frac{1}{2}\right) - 1 \quad \text{(using $x = \frac{1}{2}u + 1$)} \\
    &= 4(u_{n+1})\left[\frac{1}{2} - \frac{1}{4}u_{n+1}\right] - 1 \\
    &= 1 - 2u_{n+1}^2.
\end{align*}
\]

Thus $f(x) = 4x(1-x)$ on $[0, 1]$ becomes $g(u) = 1 - 2u^2$ on $[-1, 1]$. 

More theoretically, let $x_{n+1} = f(x_n)$ be a dynamical system, and let $u = \phi(x)$ be a change of variable. That is, invertible mapping of the state space of $f$. Then

\[
\begin{align*}
    u_{n+1} &= \phi(x_{n+1}) \\
    &= \phi(f(x_n)) \\
    &= \phi(f(\phi^{-1}(u_n))).
\end{align*}
\]

So $x_{n+1} = f(x_n)$ becomes

\[
    u_{n+1} = g(u_n),
\]

where $g(u) = \phi(f(\phi^{-1}(u)))$. That is,

\[
    g = \phi \circ f \circ \phi^{-1}.
\]  

Diagram:

\[
\begin{array}{ccc}
    S & \overset{f}{\rightarrow} & S \\
    \downarrow \phi & & \downarrow \phi \\
    T & \overset{g}{\rightarrow} & T
\end{array}
\]

Here $S$ is the state space for $f$ and $T = \phi(S)$ is that for $g$. If $f$ and $g$ are related by equation (1) we say they are conjugate and that $\phi$ is a conjugacy.

Lemma 2.9 If $f$ and $g$ are conjugate, then so are $f^k$ and $g^k$. Indeed,

\[
g = \phi \circ f \circ \phi^{-1} \implies g^k = \phi \circ f^k \circ \phi^{-1}.
\]

Remark: This should be familiar from linear algebra, with similar matrices: if $A = PBP^{-1}$ then $A^k = PB^kP^{-1}$ (and the proof is essentially the same).

Proof By induction: for $k = 1$ this is the hypothesis. Suppose then that the statement is true for $k = r$; that is, $g^r = \phi \circ f^r \circ \phi^{-1}$. Then

\[
\begin{align*}
    g^{r+1} &= g \circ g^r = (\phi \circ f \circ \phi^{-1}) \circ (\phi \circ f^r \circ \phi^{-1}) \\
    &= \phi \circ f \circ f^r \circ \phi^{-1} \quad \text{(as $\phi^{-1} \circ \phi = \text{identity}$)} \\
    &= \phi \circ (f \circ f^r) \circ \phi^{-1},
\end{align*}
\]

so the statement is true for $k = r + 1$, as required. 

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2.7.1 Common properties of conjugate maps

Throughout assume \( f : S \to S \) and \( g : T \to T \) are conjugate via \( \phi \). So \( g = \phi \circ f \circ \phi^{-1} \) or equivalently \( f = \phi^{-1} \circ g \circ \phi \).

**Fixed points** If \( p \in S \) is a fixed point of \( f \) then \( q = \phi(p) \in T \) is a fixed point of \( g \).

*Proof:* We want to show \( g(q) = q \). We have (since \( p = \phi^{-1}(q) \)),
\[
  g(q) = \phi \circ f \circ \phi^{-1}(q) = \phi(f(p)) = \phi(p) = q.
\]
□

**Periodic points** If \( p \in S \) is a periodic point of \( f \) of period \( n \) then \( q = \phi(p) \in T \) is a periodic point of \( g \) of period \( n \).

This follows immediately by Lemma 2.9 and the result above for fixed points [Exercise: supply the details]. □

**Stability** Suppose \( \phi \) is a *homeomorphism* (meaning both \( \phi \) and \( \phi^{-1} \) are continuous). Then \( p \) is an attracting (respectively repelling) periodic point of \( f \) iff \( q = \phi(p) \) is an attracting (resp. repelling) periodic point of \( g \).

(Of course, ‘periodic point’ in this statement includes ‘fixed point’!)

*Proof:* Just prove the ‘attracting’ part. Now \( p \) is an attracting periodic point of \( f \) of periodic \( k \) if and only if it is an attracting fixed point of \( f^k \). Since \( f^k \) and \( g^k \) are also conjugate by \( \phi \) (Lemma 2.9), we will assume \( p \) is a fixed point of \( f \) (it suffices to replace \( f \) by \( f^n \) and \( g \) by \( g^n \) throughout to prove the statement about periodic points).

Since \( p \) is an attracting fixed point of \( f \), there is a neighbourhood \( U \) of \( p \) in \( S \) such that \( x_0 \in U \Rightarrow x_n \to p \) (here \( x_n = f^n(x_0) \) is the orbit of \( x_0 \) as usual).

Let \( V = \phi(U) \subset T \). Then \( V \) is open in \( T \) and contains \( q \) (open because \( V = (\phi^{-1})^{-1}(U) \), and the inverse image of an open set is open (using that \( \phi^{-1} \) is open)).

Let \( y_0 \in V \), and let \( x_0 = \phi^{-1}(y_0) \in U \). Then \( x_n \to p \). Since \( \phi \) is continuous, it follows that \( \phi(x_n) \to \phi(p) = q \) (because \( \phi \) is continuous).

Finally we show that \( y_n = \phi(x_n) \). By definition,
\[
y_n = g^n(y_0) = \phi \circ f^n \circ \phi^{-1}(y_0) = \phi(f^n(x_0)) = \phi(x_n),
\]
as required. □

2.7.2 Further properties of topological conjugacy

If \( \phi \) is a homeomorphism, we often say it defines a *topological conjugacy*.

If \( f \) and \( g \) are topologically conjugate and if \( f \) has a dense orbit then so does \( g \). Moreover if \( f \) exhibits SDIC at \( x \) then \( g \) exhibits SDIC at \( y = \phi(x) \).

Thus, if \( f \) is chaotic, so is \( g \).

We leave these without proof. The density statement is not hard to prove, but the SDIC one is not so straightforward.
2.7.3 Example

We have seen that the tent and logistic maps have many properties in common. This is because they are conjugate: let $F(u) = 4u(1-u)$ and $T(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 2-2x & \text{if } x \geq 1/2 \end{cases}$. Both define dynamical systems on $[0,1]$.

The conjugacy is given by $u = \phi(x) = \frac{1}{2}(1 - \cos \pi x)$. NB: $\phi$ and $\phi^{-1}$ are both continuous (ie, $\phi$ is a homeomorphism), even though $\phi^{-1}$ is not differentiable at $u = 0, 1$.

To show that $F \circ \phi = \phi \circ T$ is just a calculation:

$$F \circ \phi(x) = F\left(\frac{1}{2}(1 - \cos \pi x)\right) = 4 \frac{1}{2}(1 - \cos \pi x)(1 - \frac{1}{2}(1 - \cos \pi x))$$

$$= (1 - \cos \pi x)(1 + \cos \pi x)$$

$$= 1 - \cos^2 \pi x = \sin^2 \pi x.$$

On the other hand,

$$\phi \circ T(x) = \begin{cases} \phi(2x) & \text{if } x \leq 1/2 \\ \phi(2 - 2x) & \text{if } x \geq 1/2 \end{cases}$$

$$= \begin{cases} \frac{1}{2}(1 - \cos 2\pi x) & \text{if } x \leq 1/2 \\ \frac{1}{2}(1 - \cos(2\pi - 2\pi x)) & \text{if } x \geq 1/2 \end{cases}$$

$$= \frac{1}{2}\cos 2\pi x = \sin^2 \pi x.$$

Thus $F \circ \phi = \phi \circ T$ so the maps are indeed conjugate.

2.7.4 Logistic map has contracting transition graph

Let $A_1 A_2 \ldots A_k$ be a sequence of Ls and Rs (an allowed path in the transition graph of the logistic map), and let $J$ be the corresponding interval in $[0,1]$.

Then $\text{length}(J) \leq \frac{1}{2^k}$. 

Proof: Uses the conjugacy from the example above.

Let $I$ be the corresponding interval for the tent map. Then we (you) have shown that $\text{length}(I) \leq \frac{1}{2^k}$.

Write $I = [x_1, x_2]$. Then $J = [u_1, u_2]$ with $u_1 = \phi(x_1)$ and $u_2 = \phi(x_2)$. Then

$$u_2 - u_1 = \phi(x_2) - \phi(x_1) = \int_{x_1}^{x_2} \phi'(x) \, dx$$

$$= \frac{\pi}{2} \int_{x_1}^{x_2} \sin \pi x \, dx$$

$$\leq \frac{\pi}{2} \int_{x_1}^{x_2} 1 \, dx$$

$$= \frac{\pi}{2} (x_2 - x_1) = \frac{\pi}{2} \cdot \frac{1}{2^k}$$

$$< 4 \cdot \frac{1}{2^k} = \frac{1}{2^{k-1}},$$

as required.