0C1/1C1 January 2014 Solutions

1.

(1)(i) $(x^2-4)(x+3) = x^2(x+3) - 4(x+3) = x^3 + 3x^2 - 4x - 12$

(1)(ii) $(a-b-2)(a+b-1) = a(a+b-1) - b(a+b-1) - 2(a+b-1)$
$= a^2 + ab - a - ba - b^2 + b - 2a - 2b + 1 = a^2 - b^2 - 3a - b + 2$

(1)(iii) $(2-x)(1-(x-1)) = (2-x)(1-x+1) = (2-x)(2-x) = x^2 - 4x + 4$

(1)(iv) $(1-x)(2x-1)^2 = (1-x)(2x-1)(2x-1) = (1-x)(1-4x+4x^2)$
$= 1 - 4x + 4x^2 - x + 4x^2 - 4x^3 = -4x^3 + 8x^2 - 5x + 1.$

(2) In 1(iv) the term in x is $-5x$, the coefficient of x^3 is -4 and the constant term is 1.

(3)

(i) $\dfrac{x^2}{x^5} = x^{2-5} = x^{-3}$ (ii) $x^{-1}\sqrt[4]{x} = x^{-1+1/4} = x^{-3/4}$

(iii) $(x^5)^{3/10} = x^{5\times 3/10} = x^{15/10} = x^{3/2}$

2. (1) $x^2 - 7x + 10 = (x-2)(x-5)$ so $x^2 - 7x + 10 = 0 \iff x = 2, 5.$
Or use the formula to give solutions

$$\frac{-(-7) \pm \sqrt{(-7)^2 - 4.10}}{2} = \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2} = 2, 5.$$

(2)

$$3x^2 + x - 2 = -x^2 + 2x - 1 \iff 4x^2 - x - 1 = 0 \iff x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.4.(-1)}}{8}$$

$$\iff x = \frac{1 \pm \sqrt{17}}{8}.$$

(3)

$$\frac{x+6}{x-2} = \frac{x-9}{3} \iff 3(x+6) = (x-2)(x-9) \iff 3x + 18 = x^2 - 11x + 18$$

$$\iff x^2 - 14x = 0 \iff x(x-14) = 0 \iff x = 0, 14.$$

(4)

$$\frac{2}{x}-\frac{1}{x-2}=\frac{1}{x-6} \iff 2(x-2)(x-6)-x(x-6) = x(x-2)$$
$$\iff 2x^2-16x+24-x^2+6x = x^2-2x$$
$$\iff -8x+24 = 0$$
$$\iff x=3$$

(5) Put $y=x^3$, so the equation becomes

$$y^2-7y-8=0 \iff (y+1)(y-8)=0 \iff y=-1,8$$

Hence $x^3=y=-1,8$ so $x=\sqrt[3]{-1},\sqrt[3]{8}=-1,2$ are the solutions.

3.
(1) $16^x=4 \iff (4^2)^x=4 \iff 4^{2x}=4 \iff 2x=1 \iff x=1/2.$

(2) $\log_2\left(\frac{2}{x-1}\right)=-3 \iff \left(\frac{2}{x-1}\right)=2^{-3} \iff \frac{2}{x-1}=\frac{1}{8}$
$\iff x-1=16 \iff x=17$

(3) $\log_3\left(9^{x+1}\right)=3x \iff (x+1)\log_3(9)=3x$
$\iff 2(x+1)=3x \iff x=2.$

(4) $x\log_x(4)=\log_x(3) \iff x=\frac{\log_x(4)}{\log_x(3)} \iff x=\log_4(3)$

(5) $\log_x(x-1)+\log_x(3)=1 \iff \log_x(3(x-1))=1$
$\iff 3(x-1)=x^1=x \iff x=3/2$

4. (1) If $y=mx+c$ passes through both $(-2,14)$ and $(3,4)$ then

$$14=-2m+c \quad \text{and} \quad 4=3m+c.$$

Subtracting the second equation from the second gives $10=-5m$ so $m=-2$ and with the second equation this gives $4=-6+c$ so $c=10$ and the equation of the line $\mathcal{C}$ is $y=-2x+10$.

(2) If the equation of line D is $y=mx+c$ then we must have $m=-2$, since being parallel to C it has the same slope. Also since it passes through the point $(2,1)$ we must have $1=2m+c=2(-2)+c$ so $c=5$ and the equation of D is $y=-2x+5$.

(3) If the equation of line E is $y=mx+c$ then we must have $m=1/2$, since being perpendicular to C we must have $m(-2)=-1$. Also since it passes through the point $(2,1)$ we must have $1=2m+c=2(1/2)+c$ so $c=0$ and the equation of D is $y=x/2$.

(4) If the lines $y=-2x+10$ and $y=x/2$ intersect at (x,y) then

$$-2x+10=y=x/2$$

so $5x/2=10$, i.e. $x=4$ and $y=(4)(1/2)=2$. So the required point is $(4,2)$.

(5) The distance between the lines C,D is the distance between the points $(2,1)$ and $(4,2)$, in other words

$$\sqrt{(2-4)^2+(1-2)^2}=\sqrt{4+1}=\sqrt{5}.$$

5.

(1) At the point of intersection we must have $-x^2 + 6x - 7 = y = x^2 - 2x - 1$. Hence $0 = 2x^2 - 8x + 6 = 2(x-3)(x-1)$ so $x = 3, 1$. For these values of x, the values of y are respectively $y = 3^2 - 2.3 - 1 = 2$ and $y = (1)^2 - 2.(1) - 1 = -2$. Hence the required points are $(3, 2), (1, -2)$.

(2) Substituting $x = -1, y = 2$ into $y = x^2 - 2x - 1$ gives $2 = (-1)^2 - 2) - 1) - 1$ which balances. Hence this point $(-1, 2)$ is on the curve $\mathcal{D}$. At this point the slope of $\mathcal{D}$ is $2(-1) - 2 = -4$. Letting the tangent here be $y = mx + c$ we therefore must have $m = -4$ and since this tangent goes through the point $(-1, 2)$, $2 = m(-1) + c = (-4)(-1) + c$, so $c = -2$ and the required tangent is $y = -4x - 2$.

(3) The slopes of C, D respectively are

$$\frac{d}{dx}(-x^2 + 6x - 7) = -2x + 6, \qquad \frac{d}{dx}(x^2 - 2x - 1) = 2x - 2,$$

so these are equal when $-2x + 6 = 2x - 2$, i.e. $x = 2$. At this point $\mathcal{D}$ (and $\mathcal{C}$) have slope $2.2 - 2 = 2$. The curve $y = ce^x$ has slope

$$\frac{d}{dx}(ce^x) = ce^x$$

at x so at $x = 2$ it has slope ce^2. Therefore for this to equal 2 we must have $ce^2 = 2$, i.e. $c = 2e^{-2}$.

6. (1)(i) The domain of f is all the real numbers except -1.

(ii)

$$f(f(x)) = \frac{1}{1 + f(x)} - 1 = \frac{1}{1 + \frac{1}{1+x} - 1} - 1 = 1 + x - 1 = x.$$

(iii) Either notice from (2) above that

$$f^{-1}(x) = f(x) = \frac{1}{1+x} - 1$$

or solve $f(y) = x$ to give $y = f^{-1}(x)$, i.e.

$$\begin{aligned} f(y) = x \quad &\iff \quad \frac{1}{1+y} - 1 = x \\ &\iff \quad \frac{1}{1+y} = 1 + x \\ &\iff \quad 1 = (1+y)(1+x) \\ &\iff \quad \frac{1}{1+x} = 1 + y \\ &\iff \quad y = \frac{1}{1+x} - 1 \ \left(= \frac{-x}{1+x}\right) \end{aligned}$$

(2) (i) $\cos(A) = b/6$ so $b = 6\cos(A) = 6 \times (1/3) = 2$.

(ii) $\sin^2(A) + \cos^2(A) = 1$ so $\sin^2(A) = 1 - (1/3)^2 = 8/9$ and $\sin(A) = \sqrt{8/9} = 2\sqrt{2}/3$. [From the diagram $\sin(A) \geq 0$ so we must take the positive square root here.]

(iii) $\tan(A) = \sin(A)/\cos(A) = (2\sqrt{2}/3)/(1/3) = (2\sqrt{2}/3) \times 3 = 2\sqrt{2}$

(iv) $\sin(2A) = 2\sin(A)\cos(A) = 2 \times (2\sqrt{2}/3) \times (1/3) = 4\sqrt{2}/9$

(v) $\cos(A - \pi/2)) = \cos(A)\cos(\pi/2) + \sin(A)\sin(\pi/2) = (1/3) \times 0 + (2\sqrt{2}/3) \times 1 = (2\sqrt{2}/3)$.

7.(1)

(i) $d/dx(5x^6 - 6) = 30x^5$

(ii) $d/dx(x^{-1/4}) = (-1/4) \times x^{-1-1/4} = \dfrac{-x^{-5/4}}{4}$

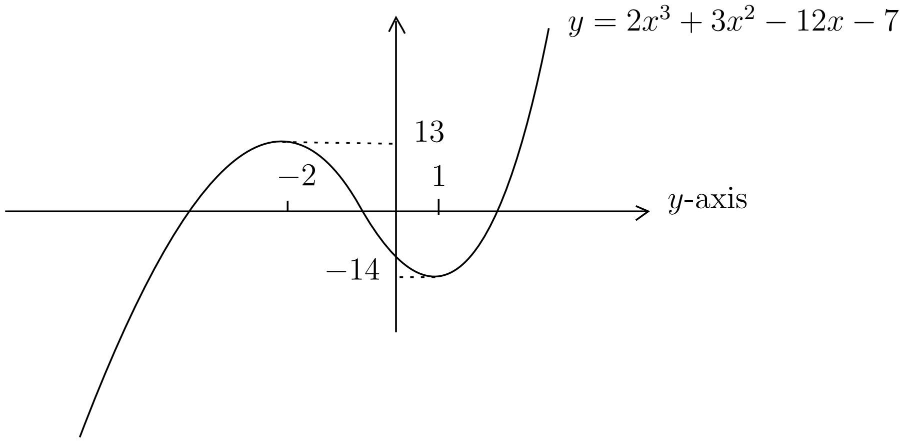
(iii) $d/dx(e^{2x+1}) = 2e^{2x+1}$

(2) For $f(x) = 2x^3 + 3x^2 - 12x - 7$, $df/dx = 6x^2 + 6x - 12$ and $d^2f/dx^2 = 12x + 6$. Hence the stationary points are when

$$df/dx = 6x^2 + 6x - 12 = 6(x+2)(x-1) = 0,$$

i.e. $x = -2$, 1. When $x = -2$ $d^2f/dx^2 = 12(-2) + 6 = -18 < 0$ so this is a (local) maximum. When $x = 1$, $d^2f/dx^2 = 12(1) + 6 = 18 > 0$ so this is a (local) minimum point.

The graph of this function looks as below.



In other words it goes up from the left to reach a local maximum at $x = -2$, when in fact $f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) - 7 = 13$. It then goes down to a local minimum at $x = 1$ (when $f(1) = 2(1)^3 + 3(1)^2 - 12(1) - 7 = -14$) and thereafter moves up to the right. A solution of $2x^3 + 3x^2 - 12x - 7 = 0$ is a value of x for which this curve crosses the y-axis (i.e. the line $y = 0$) and since this happens 3 times there must be 3 solutions to this equation.

8.

(1) $\dfrac{d}{dx}(2x+1)^{-2} = (2)(-2)(2x+1)^{-2-1} = -4(2x+1)^{-3}$

(2) $\dfrac{d}{dx}(\sin^2(x)) = \dfrac{d}{dx}(\sin(x)) \cdot \sin(x) + \sin(x) \cdot \dfrac{d}{dx}(\sin(x))$

$= (\cos(x))\sin(x) + \sin(x)(\cos(x)) = 2\cos(x)\sin(x)$

(3) $\dfrac{d}{dx}\left(\dfrac{x}{1-2x}\right) = \dfrac{(1)(1-2x)-(-2)x}{(1-2x)^2} = = \dfrac{1}{(1-2x)^2}$

(4) Putting $u = 2+\cos(x)$, $y = \ln u$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (-\sin(x)) = \frac{-\sin(x)}{2+\cos(x)}$$

(5) Putting $u = 1+e^x$, $y = \sqrt[3]{u}$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{u^{-2/3}}{3} \cdot (e^x) = \frac{e^x}{3(1+e^x)^{2/3}}$$

Feedback

Generally the exam was very well done with a lot of students scoring over 50 out of the possible 60 and very few students scoring abysmally. Strangely 1C1 students concentrated mainly on questions 1-5 whereas 0C1 students spread their choices much more evenly. As usual most marks were lost through silly arithmetic errors rather then not knowing what to do. In particular some students still struggle to add and subtract fractions. Some comments below on the individual questions:

Q1 Very popular as usual with most students scoring well. The commonist errors (apart from simple arithmetic) were in not knowing what term and coefficient meant in part (2) and in not converting the answer $x^{15/10}$ in (3)(iii) to its simplest form $x^{3/2}$ as you were asked to do.

Q2 Well done, even the trickier part (5). As usual some students landed up with answers containing a negative square root but didn't draw the obvious conclusion that they must have made a mistake and go back to check their working. In part (3) a quite common error was to arrive at $x(x-14) = 0$ and then conclude that $x = 14$, so missing the other solution $x = 0$. In part (4) a remarkably common error in multiplying out was to get $-x(x-6)$ (from the middle term) and then replace this by $-x^2 - 6x$ instead of the correct $-x^2 + 6x$.

Q3 Well done though a surprising number of students got to $2x = 3$ in part (5) and then concluded that $x = 2/3$!!

Q4 Generally well done except for the last part, which I imagine shows a lack of visualization. Look, the line E is perpendicular to both C and D so the distance between these lines is the distance between the points where E intersects them, ie. $(2,1)$ and $(4,2)$.

Q5 Most students struggled with the innovative last part, finding c. If you're one of them take a glance at the solutions. There was a certain amount of 'wishful thinking' on part (2) when showing that the point $(-1,2)$ is on the curve $\mathcal{D}$ – students substituted $x = -1, y = 2$ into $y = x^2 - 2x - 1$ and said it balanced correctly even if patently didn't according to their faulty arithmetic!

Q6 Few takers for this question and generally it wasn't done that well. In particular most students just seemed to guess an answer for part (i).

Q7 A popular question but not so well done. A common mistake in part (1)(ii) was to simplify $\frac{-1}{4}.x^{-5/4}$ to $\frac{x^{-5/4}}{4}$ thus losing the -1. [In examples like this I do expect you to express the answer in its simplest form, that's part of the question and failing to do so can lose you marks.] In part (2) a common error was to draw the graph of $y = 2x^3 + 3x^2 - 12x - 7$ and then draw in the line $y = 7$ and conclude that because it crossed the graph in 3 places the equation

$$2x^3 + 3x^2 - 12x - 7 = 0$$

has 3 solutions. But what this actually shows is that the equation

$$2x^3 + 3x^2 - 12x - 7 = 7$$

has 3 solutions!! What you really needed to point out was that the x-axis (i.e. the line $y = 0$) cuts the graph in 3 places.

Q8 Mostly well done, I suppose that students who felt uneasy about differentiation already steered clear of this one.