0C1/1C1 January 2013 Solutions

1.

(1)(i) $(x^2-3)(x+4) = x^2(x+4) - 3(x+4) = x^3+4x^2-3x-12$

(1)(ii) $(a-b+1)(a+b-1) = a(a+b-1) - b(a+b-1) + (a+b-1)$

$= a^2+ab-a-ba-b^2+b+a+b-1 = a^2-b^2+2b-1$

(1)(iii) $(2-x)(1-(x-2)) = (2-x)(1-x+2) = (2-x)(3-x) = 6-5x+x^2$

(1)(iv) $(1-2x)(x-1)^2 = (1-2x)(1-x)(1-x) = (1-2x)(1-2x+x^2) =$

$1-2x+x^2-2x+4x^2-2x^3 = -2x^3+5x^2-4x+1.$

(2) In 1(iv) the term in x^2 is $5x^2$, the coefficient of x is -4 and the constant term is 1.

(3)

(i) $\dfrac{x^4}{x^8} = x^{4-8} = x^{-4}$ (ii) $x^{-2}\sqrt[3]{x} = x^{-2+1/3} = x^{-5/3}$ (iii) $(x^4)^{5/6} = x^{4\times 5/6} = x^{20/6} = x^{10/3}$

2. (1) $x^2-6x+9 = (x-2)(x-4)$ so $x^2-6x+8=0 \iff x=2,4.$
Or use the formula to give solutions

$$\frac{-(-6)\pm\sqrt{(-6)^2-4.8}}{2} = \frac{6\pm\sqrt{4}}{2} = \frac{6\pm 2}{2} = 2,4.$$

(2)

$$5x^2+4x-2 = 3x^2+x-1 \iff 2x^2+3x-1=0 \iff x = \frac{-3\pm\sqrt{3^2-4.2.(-1)}}{4}$$

$$\iff x = \frac{-3\pm\sqrt{17}}{4}.$$

(3)

$$\frac{x+5}{x-5} = \frac{x-2}{2} \iff 2(x+5) = (x-5)(x-2) \iff 2x+10 = x^2-7x+10$$

$$\iff x^2-9x=0 \iff x(x-9)=0 \iff x=0,9.$$

(4)

$$\frac{2}{x+2} - \frac{1}{x} = \frac{1}{x-4} \iff 2x(x-4) - (x-4)(x+2) = x(x+2)$$
$$\iff 2x^2 - 8x - x^2 + 2x + 8 = x^2 + 2x$$
$$\iff -8x + 8 = 0$$
$$\iff x = 1$$

(5) Put $y = (x+1)^2$, so the equation becomes

$$y^2 - 5y + 4 = 0 \iff (y-4)(y-1) = 0 \iff y = 1, 4$$

Hence $(x+1)^2 = y = 1, 4$ so $x + 1 = \pm 1, \pm 2$ and $x = 0, -2, 1, -3$ are the solutions.

3.
(i) $9^x = 3 \iff (3^2)^x = 3 \iff 3^{2x} = 3^1 \iff 2x = 1 \iff x = 1/2.$

(ii) $\log_3\left(\frac{2}{x+8}\right) = -2 \iff \left(\frac{2}{x+8}\right) = 3^{-2} \iff \frac{2}{x+8} = \frac{1}{9}$
$\iff x + 8 = 18 \iff x = 10$

(iii) $\log_3\left(9^{x+1}\right) = x \iff (x+1)\log_3(9) = x$
$\iff 2(x+1) = x \iff x = -2.$

(iv) $x\log_x(3) = \log_x(2) \iff x = \frac{\log_x(2)}{\log_x(3)} \iff x = \log_3(2)$

(v) $\log_x(4x-4) = 2 \iff 4x - 4 = x^2 \iff (x-2)^2 = 0 \iff x = 2$

4. (1) If $y = mx + c$ passes through both $(-2, 1)$ and $(1, 7)$ then

$$1 = -2m + c \quad \text{and} \quad 7 = m + c.$$

Subtracting the first equation from the second gives $6 = 3m$ so $m = 2$ and with the second equation this gives $7 = 2 + c$ so $c = 5$ and the equation of the line $\mathcal{C}$ is $y = 2x + 5$.

(2) Substituting $x = 3$, $y = 10$ into $y = 2x + 5$ gives $10 = (2)(3) + 5$ which is not true so this point does not lie on the line $\mathcal{C}$.

(3) If the lines $y = 2x + 5$ and $y = 1 - 2x$ intersect at (x, y) then

$$2x + 5 = y = 1 - 2x$$

so $2x + 2x = 1 - 5$, i.e. $x = -1$ and $y = (2)(-1) + 5 = 3$. Hence $A = (-1, 3)$.

(4) The gradients of the lines $y = 2x + 5$, $y = 1 - 2x$ are $2, -2$ respectively and since $(2)(-2) \neq -1$ they are not perpendicular.

(5) The distance from $(-1, 3)$ to $(0, 5)$ is

$$\sqrt{((-1 - 0)^2 + (3 - 5)^2} = \sqrt{1 + 4} = \sqrt{5}.$$

(6) Considering the right angled triangle formed by the points $A = (-1, 3)$, $(0, 5)$ and $(0, 3)$ (notice that $(-1, 3)$ and $(0, 5)$ lie on the line $\mathcal{C}$ and the line segment from $(-1, 3)$ and $(0, 3)$ is parallel to the x-axis and of length 1) we see that the cosine of the angle $\mathcal{C}$ makes with the x-axis is the distance from $(-1, 3)$ to $(0, 3)$, divided by the distance from $(-1, 3)$ to $(0, 5)$, i.e. $1/\sqrt{5}$.

5.

(1) The curves intersect when

$$x^2 - 2 = y = 2x^2 + 7x - 2,$$

equivalently

$$0 = x^2 + 7x = x(x + 7).$$

Thus the two points are when $x = 0$ and $y = 0^2 - 2 = -2$ and when $x = -7$ and $y = (-7)^2 - 2 = 47$, i.e. the points $(0, -2)$ and $(-7, 47)$.

(2) The slopes of these curves at x are $(d/dx)(x^2 - 2) = 2x$ and $(d/dx)(2x^2 + 7x - 2) = 4x + 7$ respectively so these will be equal when $2x = 4x + 7$, i.e. $x = -7/2$.

(3) The point $(-1, -7)$ is on the curve $\mathcal{D}$ since substituting in these values into the defining equation gives $-7 = 2(-1)^2 + (-1)7 - 2$. At this value of x the slope is as above $4(-1) + 7 = 3$ so if $y = mx + c$ is to the the tangent it must satisfy that $m = 3$ and $-7 = (3)(-1) + c$ so $c = -4$, i.e. the equation of the tangent is $y = 3x - 4$.

(4) The line $y = 3x - 4$ meets the curve $\mathcal{C}$ when $3x - 4 = y = x^2 - 2$, equivalently $x^2 - 3x + 2 = 0 = (x - 2)(x - 1)$, i.e. $x = 1$ and $x = 2$. At $x = 1$

the y coordinate on $\mathcal{C}$ is $1^2 - 2 = -1$ and at $x = 2$ the y coordinate on $\mathcal{C}$ is $2^2 - 2 = 2$. Hence the two points are $(1, -1)$ and $(2, 2)$.

6. (a) (1) Domain of f is all the reals except 0.

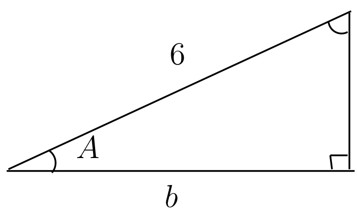
(2) $f(f(x)) = \dfrac{2}{f(x)} + 1 = 2/((2/x) + 1) + 1 = 2x/(2 + x) + 1$.

(3) We must have $x = f(f^{-1}(x)) = \dfrac{2}{f^{-1}(x)} + 1$. Multiplying both sides by $f^{-1}(x)$ gives

$$xf^{-1}(x) = 2 + f^{-1}(x)$$

so $(x - 1)f^{-1}(x) = 2$ and $f^{-1}(x) = \dfrac{2}{x - 1}$.

(b)



(1) $\cos(A) = b/6$ so $b = 6\cos(A) = 4$.

(2) $\sin^2(A) + \cos^2(A) = 1$ so $\sin^2(A) = 1 - (2/3)^2 = 5/9$ and $\sin(A) = \sqrt{5/9} = \sqrt{5}/3$.

[From the diagram $\sin(A) \geq 0$ so we must take the positive square root here.]

(3) $\cot(A) = (\cos(A)/\sin(A)) = (2/3)/(\sqrt{5}/3) = 2/\sqrt{5}$

(4) $\cos(A) = 2\cos^2(A/2) - 1$ so $\cos(A/2) = \sqrt{(\cos(A) + 1)/2} = \sqrt{((2/3) + 1)/2} = \sqrt{5/6}$

[Notice it is the positive root since from the diagram $\cos(A/2)$ is positive.]

(5) $\cos(A - \pi/4) = \cos(A)\cos(\pi/4) + \sin(A)\sin(\pi/4) = (2/3)(1/\sqrt{2}) + (\sqrt{5}/3)(1/\sqrt{2}) = (2 + \sqrt{5})/(3\sqrt{2})$.

7.(1)
(i) $d/dx(6x^6 - 6) = 36x^5$

(ii) $d/dx(x^{-4/3}) = \dfrac{-4x^{-4/3-1}}{3} = \dfrac{-4x^{-7/3}}{3}$

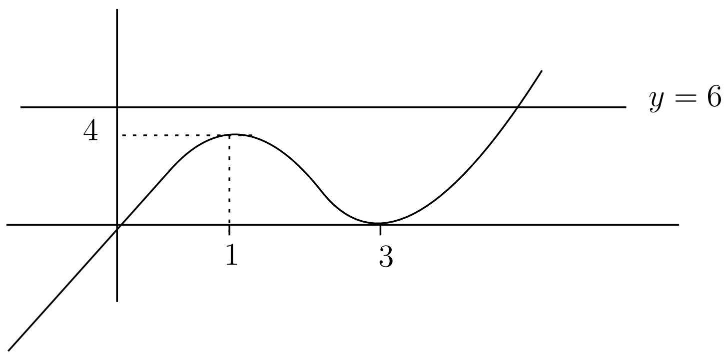
(iii) $d/dx(e^{1-2x}) = -2e^{1-2x}$

(2) For $f(x) = x^3 - 6x^2 + 9x$, $df/dx = 3x^2 - 12x + 9$ and $d^2f/dx^2 = 6x - 12$. Hence the stationary points are when

$$df/dx = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0,$$

i.e. $x = 1$, 3. When $x = 1$ $d^2f/dx^2 = 6(1) - 12 = -6 < 0$ so this is a (local) maximum. When $x = 3$, $d^2f/dx^2 = 6(3) - 12 = 6 > 0$ so this is a (local) minimum point.

The graph of this function looks as follows:



[In other words it goes up from the left to reach a local maximum at $x = 1$, when in fact $f(1) = (1)^3 - 6(1)^2 + 9(1) = 4$. It then goes down to a local minimum at $x = 3$ (when $f(3) = 0$) and thereafter moves up to the right.] Since the line $y = 6$ lies above the curve at the local maximum this line only cuts the curve (beyond $x = 3$) in one place so $x^3 - 6x^2 + 9x = 6$ has only one solution.

8.

(1) $\frac{d}{dx}(x^2+1)^{-3} = (2x)(-3)(x^2+1)^{-3-1} = -6x(x^2+1)^{-4}$

(2) $\frac{d}{dx}(\sin(x)\cos(2x)) = \frac{d}{dx}(\sin(x)) \cdot \cos(2x) + \sin(x) \cdot \frac{d}{dx}(\cos(2x))$

$= (\cos(x))\cos(2x) + \sin(x)(-2\sin(2x)) = \cos(x)\cos(2x) - 2\sin(x)\sin(2x)$

(3) $\frac{d}{dx}\left(\frac{1-x}{1+x}\right) = \frac{(-1)(1+x)-(1)(1-x)}{(1+x)^2} = = \frac{-2}{(1+x)^2}$

(4) Putting $u = 2+\sin(x)$, $y = \ln u$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (\cos(x)) = \frac{\cos(x)}{2+\sin(x)}$$

(5) Putting $u = \sqrt{x}$, $y = 2e^u$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2e^u \cdot (\frac{1}{2\sqrt{u}}) = \frac{e^{\sqrt{x}}}{\sqrt{x}}$$