0C1/1C1 January 2012 Solutions

1.

(1)(i) $(x^2-3)(x+5) = x^2(x+5) - 3(x+5) = x^3 + 5x^2 - 3x - 15$

(1)(ii) $(a-b+2)(a+b-2) = a(a+b-2) - b(a+b-2) + 2(a+b-2)$
$= a^2 + ab - 2a - ba - b^2 + 2b + 2a + 2b - 4 = a^2 - b^2 + 4b - 4$

(1)(iii) $(2-x)(2-(x-1)) = (2-x)(2-x+1) = (2-x)(3-x) = 6 - 5x + x^2$

(1)(iv) $x(1-2x)(x-1) = x(x-1-2x^2+2x) = x(-2x^2+3x-1) = -2x^3+3x^2-x.$

(2) In 1(iv) the term in x^2 is $3x^2$, the coefficient of x is -1 and the constant term is 0.

(3)

(i) $\dfrac{x^3}{x^6} = x^{3-6} = x^{-3}$ (ii) $x^{-1}\sqrt[4]{x} = x^{-1+1/4} = x^{-3/4}$ (iii) $(x^6)^{1/4} = x^{6\times 1/4} = x^{6/4} = x^{3/2}$

2. (1) $x^2 - 3x - 10 = (x-5)(x+2)$ so $x^2 - 3x - 10 = 0 \iff x = -2, 5.$
Or use the formula to give solutions

$$\frac{-(-3) \pm \sqrt{(-3)^2 - 4.(-10)}}{2} = \frac{3 \pm \sqrt{49}}{2} = \frac{3 \pm 7}{2} = -2, 5.$$

(2)

$$3x^2 + 4x - 2 = x^2 + x - 1 \iff 2x^2 + 3x - 1 = 0 \iff x = \frac{-3 \pm \sqrt{(-3)^2 - 4.2.(-1)}}{4}$$
$$\iff x = \frac{-3 \pm \sqrt{17}}{4}.$$

(3)

$$\frac{x+2}{x-4} = \frac{x-1}{2} \iff 2(x+2) = (x-4)(x-1) \iff 2x + 4 = x^2 - 5x + 4$$
$$\iff x^2 - 7x = 0 \iff x(x-7) = 0 \iff x = 0, 7.$$

(4)

$$\frac{2}{x+6} - \frac{1}{x+4} = \frac{1}{x} \iff 2x(x+4) - x(x+6) = (x+6)(x+4)$$
$$\iff 2x^2 + 8x - x^2 - 6x = x^2 + 10x + 24$$
$$\iff 8x + 24 = 0$$
$$\iff x = -3$$

(5) Put $y = (x^2+1)$, so the equation becomes

$$y^2 - 5y + 6 = 0 \iff (y-2)(y-3) = 0 \iff y = 2, 3$$

Hence $x^2 + 1 = y = 2, 3$ so $x^2 = 1, 2$ and $x = 1, -1, \sqrt{2}, -\sqrt{2}$.

3.

(i) $16^x = 4 \iff (4^2)^x = 4 \iff 4^{2x} = 4 \iff 2x = 1 \iff x = 1/2$.

(ii) $\log_3\left(\frac{2}{x-3}\right) = -1 \iff \left(\frac{2}{x-3}\right) = 3^{-1} \iff \frac{2}{x-3} = \frac{1}{3}$
$\iff x - 3 = 6 \iff x = 9$

(iii) $\log_2 (4^{x-1}) = x + 1 \iff (x-1)\log_2(4) = x + 1$
$\iff 2(x-1) = x + 1 \iff x = 3$.

(iv) $x \log_x(2) = \log_x(3) \iff x = \frac{\log_x(3)}{\log_x(2)} \iff x = \log_2(3)$

(v) $\log_x(x^3 + x - 11) = 3 \iff x^3 + x - 11 = x^3 \iff x - 11 = 0 \iff x = 11$

4. (1) If $y = mx + c$ passes through both $(-1, 2)$ and $(1, 6)$ then

$$2 = -m + c \quad \text{and} \quad 6 = m + c.$$

Subtracting the first equation from the second gives $4 = 2m$ so $m = 2$ and with the second equation this gives $6 = (2)(1) + c$ so $c = 4$ and the equation of the line $\mathcal{C}$ is $y = 2x + 4$.

(2) Substituting $x = 2$, $y = 8$ into $y = 2x + 4$ gives $8 = (2)(2) + 4$ which is true so this point lies on the line $\mathcal{C}$.

(3) If $y = 0$ then $0 = 2x + 4$ and $x = -2$ so $\mathcal{C}$ crosses the x-axis at the point $A = (-2, 0)$.

(4) The distance from $(-2, 0)$ to $(1, 6)$ is

$$\sqrt{((-2-1)^2 + (0-6)^2} = \sqrt{9 + 36} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}.$$

(5) Considering the right angled triangle formed by the points $A = (-2, 0)$, $(1, 6)$ and $(1, 0)$ (notice that $(-2, 0)$ and $(1, 6)$ lie on the line $\mathcal{C}$ and $(-2, 0)$ and $(1, 0)$ lie on the x-axis) we see that the sine of the angle $\mathcal{C}$ makes with the x-axis is the distance from $(1, 6)$ to $(1, 0)$, divided by the distance from $(-2, 0)$ to $(1, 6)$, i.e. $6/3\sqrt{5} = 2/\sqrt{5}$.

(6) If the lines $y = 2x + 4$ and $y = 16 - 2x$ intersect at (x, y) then

$$2x + 4 = y = 16 - 2x$$

so $2x + 2x = 16 - 4$, i.e. $x = 3$ and $y = (2)(3) + 4 = 10$. Hence the two lines intersect at the point $(3, 10)$.

5.

(1) At the point of intersection we must have $x^2 - x - 3 = y = x - 4$. Hence $0 = x^2 - 2x + 1 = (x-1)(x-1)$, so $x = 1$. For this value of x, $y = 1 - 4 = -3$ so the required point A is $(1, -3)$.

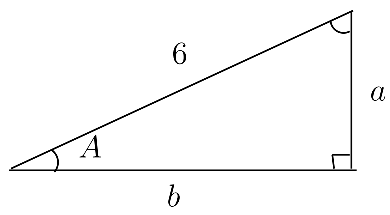
(2) At A the slope of $\mathcal{E}$ is 1 whilst the slope of $\mathcal{C}$ is $d/dx(x^2 - x - 3) = 2x - 1$ evaluated at $x = 1$, i.e. 1 again. So $\mathcal{C}$ and $\mathcal{E}$ are parallel at $(1, -3)$ and $\mathcal{E}$ must be the tangent to $\mathcal{C}$ at this point.

(3) Let the normal be $y = mx + c$. Since it is normal to $\mathcal{E}$ its slope m must satisfy $m(1) = -1$ so $m = -1$. Also since this line goes through the point $(1, -3)$, $-3 = 1(-1) + c$. Hence $c = -2$ and the normal is $y = -x - 2$.

(4) The normal intersects $\mathcal{C}$ when $x^2 - x - 3 = y = -x - 2$. Hence $0 = x^2 - 1$ so $x = 1, -1$. The solution $x = 1$ gives the point A so the other point on intersection must be when $x = -1$, and $y = -1(-1) - 2 = -1$, i.e. the point $(-1, -1)$.

(5) The slope of $\mathcal{C}$ at $x = -1$ is given by $d/dx(x^2 - x - 3) = 2x - 1$ evaluated at $x = -1$, i.e. -3. Hence the required tangent $y = mx + c$ to $\mathcal{C}$ at $(-1, -1)$ satisfies $m = -3$, $-1 = (-3)(-1) + c$, so $c = -4$ and the tangent here is $y = -3x - 4$.

6. (a)



(1) $\cos(A) = b/6$ so $b = 6\cos(A) = 9/2$.

(2) $\sin^2(A) + \cos^2(A) = 1$ so $\sin^2(A) = 1 - (3/4)^2 = 7/16$ and $\sin(A) = \sqrt{7/16} = \sqrt{7}/4$.
[From the diagram $\sin(A) \geq 0$ so we must take the positive square root here.]

(3) $a/6 = \sin(A)$ so $a = 6\sin(A) = 6\sqrt{7}/4 = 3\sqrt{7}/2$.

(4) $\tan(A) = a/b = (3\sqrt{7}/2) \times (2/9) = \sqrt{7}/3$

(5) $\cos(-A) = \cos(A) = 3/4$

(6) $\cos(2A) = \cos^2(A) - \sin^2(A) = (3/4)^2 - (\sqrt{7}/4)^2 = 9/16 - 7/16 = 1/8$

(7) $\sin(2A) = 2\sin(A)\cos(A) = 2 \times (\sqrt{7}/4) \times (3/4) = 3\sqrt{7}/8$

(8) $\cos(A) = 2\cos^2(A/2) - 1$ so $\cos(A/2) = \sqrt{(\cos(A) + 1)/2} = \sqrt{((3/4) + 1)/2} = \sqrt{7/8}$
[Notice it is the positive root since from the diagram $\cos(A/2)$ is positive.]

(9) $\cos(3A) = \cos(2A)\cos(A) - \sin(2A)\sin(A)$ so

$$\cos(3A) = (1/8) \times (3/4) - (3\sqrt{7}/8) \times (\sqrt{7}/4) = (3/32) - (21/32) = -18/32 = -9/16.$$

(10) $\sin(A + \pi/4) = \sin(A)\cos(\pi/4) + \cos(A)\sin(\pi/4) = (\sqrt{7}/4)/\sqrt{2} + (3/4)/\sqrt{2}$
$= (3 + \sqrt{7})/(4\sqrt{2})$

7.(1)

(i) $d/dx(2x^9 - 9) = 18x^8$

(ii) $d/dx(x^{1/4}) = \dfrac{x^{-1+1/4}}{4} = \dfrac{x^{-3/4}}{4}$

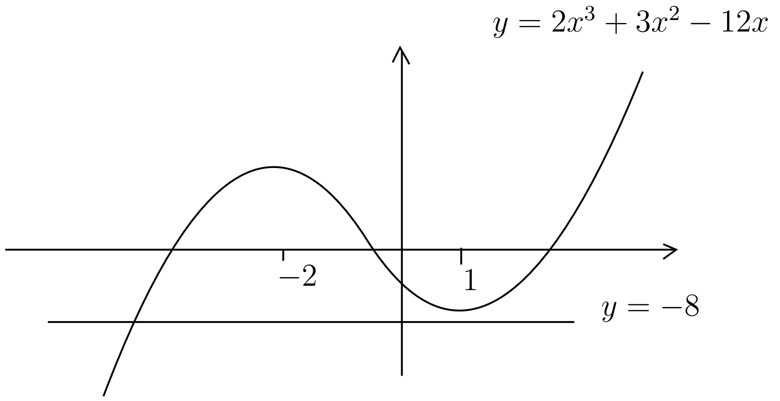
(iii) $d/dx(e^{2x+1}) = 2e^{2x+1}$

(2) For $f(x) = 2x^3 + 3x^2 - 12x$, $df/dx = 6x^2 + 6x - 12$ and $d^2f/dx^2 = 12x + 6$. Hence the stationary points are when

$$df/dx = 6x^2 + 6x - 12 = 6(x+2)(x-1) = 0,$$

i.e. $x = -2, 1$. When $x = -2$ $d^2f/dx^2 = 12(-2) + 6 = -18 < 0$ so this is a (local) maximum. When $x = 1$, $d^2f/dx^2 = 12(1) + 6 = 18 > 0$ so this is a (local) minimum point.

The graph of this function looks as below.



In other words it goes up from the left to reach a local maximum at $x = -2$, when in fact $f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) = 20$. It then goes down to a local minimum at $x = 1$ (when $f(1) = 2(1)^3 + 3(1)^2 - 12(1) = -7$) and thereafter moves up to the right. Since the line $y = -8$ lies below the curve at the local minimum this line only cuts the curve (to the left of $x = 1$) in one place so $2x^3 + 3x^2 - 12x = -8$, equivalently $2x^3 + 3x^2 - 12x + 8 = 0$, has only one solution.

8.

(1) $\dfrac{d}{dx}(2x+1)^{-2} = (2)(-2)(2x+1)^{-2-1} = -4(2x+1)^{-3}$

(2) $\dfrac{d}{dx}(\sin^2(x)) = \dfrac{d}{dx}(\sin(x)) \cdot \sin(x) + \sin(x) \cdot \dfrac{d}{dx}(\sin(x))$

$= (\cos(x))\sin(x) + \sin(x)(\cos(x)) = 2\cos(x)\sin(x)$

(3) $\dfrac{d}{dx}\left(\dfrac{1+x}{1-x}\right) = \dfrac{(1)(1-x)-(-1)(1+x)}{(1-x)^2} == \dfrac{2}{(1-x)^2}$

(4) Putting $u = 1 + e^x$, $y = \ln u$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (e^x) = \frac{e^x}{1+e^x}$$

(5) Putting $u = \cos(x) + 1$, $y = \sqrt{u}$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot (-\sin(x)) = -\frac{\sin(x)}{2\sqrt{\cos(x)+1}}$$