Two and a half hours

THE UNIVERSITY OF MANCHESTER

PREDICATE LOGIC

14th January 2013

9.45 – 12.15

Answer ALL FIVE questions in Section A (56 marks in total).
Answer TWO of the THREE questions in Section B (24 marks in total).

If more than TWO questions from Section B are attempted
then credit will be given for the FIRST TWO answers.

A list of axioms and rules of proof is appended to this examination paper

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The use of calculators is not permitted

SECTION A

Answer all FIVE questions

A1. Let the language L have a binary relation symbol R and a unary function symbol f.

Which of the following are terms of L? You should briefly justify your answers.

(i) $f(w_1)$

(ii) $f)x_1($

Which of the following are formulae of L? You should briefly justify your answers.

(iii) $\exists w_2(R(w_2, x_1) \to \forall w_1 R(w_1, x_1))$

(iv) $(\neg\exists w_1\, R(x_1, x_1))$

Let M be the structure for L with $|M| = \mathbb{N}^+ = \{1, 2, 3, \ldots\}$, $f^M(n) = n + 1$,

$$R^M = \{\langle n, m\rangle \in |M|^2 \mid n \text{ divides } m\}.$$

Which of the following sentences of L are true in M?

(v) $\forall w_1\, R(w_1, f(w_1))$

(vi) $\forall w_1 \exists w_2\, (R(w_1, w_2) \wedge \neg R(w_2, w_1))$

(vii) $\exists w_1 \forall w_2 \forall w_3\, ((R(w_2, f(w_1)) \wedge R(w_3, f(w_1))) \to (R(w_2, w_3) \vee R(w_3, w_2)))$

Find formulae $\theta_1(x_1)$, $\theta_2(x_1, x_2)$, $\theta_3(x_1)$, $\theta_4(x_1)$ of L such that for $n, m \in |M|$,

(viii) $M \models \theta_1(n) \iff n = 1$

(ix) $M \models \theta_2(n, m) \iff n = m$

(x) $M \models \theta_3(n) \iff n$ is odd

(xi) $M \models \theta_4(n) \iff n$ is a power of 2

Let K be the structure for L with $|K| = |M| = \mathbb{N}^+$, $f^K(n) = f^M(n) = n + 1$,

$$R^K = \{\langle n, m\rangle \in |K|^2 \mid n \le m\}.$$

(xii) Find a sentence ϕ of L such that $K \models \phi$ and $M \not\models \phi$.

[23 marks]

A2. Write down two different sentences in Prenex Normal Form each logically equivalent to

$$(\forall w_1 \exists w_2\, R(w_1, w_2) \to \exists w_1 R(w_1, w_1)),$$

where R is a binary relation symbol.

[5 marks]

A3. Define what is meant by a *formal proof*. Give a formal proof of

$$\exists w_1\, P(w_1),\ \forall w_1\, (P(w_1) \to Q(w_1)) \vdash \exists w_1\, Q(w_1)$$

where P, Q are unary relation symbols.

[9 marks]

A4. State the Completeness Theorem for Relational Languages. Using this theorem or otherwise show that

(a) $\exists w_1 \forall w_2\, (R(w_1, w_2) \vee R(w_2, w_1)) \vdash \exists w_1\, R(w_1, w_1)$

(b) $\forall w_1 \exists w_2\, R(w_1, w_2) \nvdash \exists w_1\, R(w_1, w_1)$

where R is a binary relation symbol.

Is it the case that

$$R(x_1, x_1) \equiv R(x_2, x_2)\ \ ?$$

Briefly explain your answer.

[10 marks]

A5. Let L be the language with a unary relation symbol P and a unary function symbol f. Show that no two of the following sentences of L logically imply the third:

(i) $\forall w_1\, (P(w_1) \to P(f(w_1)))$

(ii) $\forall w_1\, (P(w_1) \vee \neg P(f(w_1)))$

(iii) $\exists w_1\, \neg P(f(w_1))$

[9 marks]

SECTION B

Answer TWO of the THREE questions

If more than TWO questions are attempted then credit will be given for the FIRST TWO answers.

B6. Let the language L have just a single unary relation symbol P and let M be a structure for L such that $|M| = \mathbb{N}$, $1 \in P^M$, $0 \notin P^M$. Define a function $q : |M| \to \{0, 1\}$ by

$$q(n) = \begin{cases} 1 & \text{if } n \in P^M, \\ 0 & \text{if } n \notin P^M. \end{cases}$$

Show by induction on $|\theta|$ that for $\theta(x_1, x_2, \ldots, x_m) \in FL$ and $n_1, n_2, \ldots, n_m \in |M|$,

$$M \models \theta(n_1, n_2, \ldots, n_m) \iff M \models \theta(q(n_1), q(n_2), \ldots, q(n_m)).$$

[For the connectives and quantifiers it is enough to do just one of the cases.]

[12 marks]

B7. Let L be the language with equality, a unary relation symbol P and a constant symbol c. Give formal proofs of:

(i) $EqL,\ \forall w_1\, c = w_1 \vdash P(c) \to \forall w_1\, P(w_1)$

(ii) $\exists w_1\, P(w_1) \vdash \exists w_1 \exists w_2\, (P(w_1) \wedge P(w_2))$

[12 marks]

B8. State the Compactness Theorem for Relational Languages.

Let L be the language with just a ternary relation symbol T. A structure M for L is said to have a *finite separation* if there is a finite subset A of $|M|$ such that for every $b, c \in |M|$, $M \models T(b, a, c)$ for some $a \in A$. Show that there can be no sentence $\theta \in SL$ such that for any structure M for L,

$$M \models \theta \iff M \text{ has a finite separation.}$$

[12 marks]

The Rules of Proof and Axiom for the Predicate Calculus

And In (AND)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \phi}{\Gamma \cup \Delta \mid \theta \wedge \phi}$	
And Out (AO)	$\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta} \qquad \dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$	
Or In (ORR)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi} \qquad \dfrac{\Gamma \mid \theta}{\Gamma \mid \phi \vee \theta}$	
Disjunction (DIS)	$\dfrac{\Gamma, \theta \mid \psi, \quad \Delta, \phi \mid \psi}{\Gamma \cup \Delta, \theta \vee \phi \mid \psi}$	
Implies In (IMR)	$\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$	
Modus Ponens (MP)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \theta \to \phi}{\Gamma \cup \Delta \mid \phi}$	
Not In (NIN)	$\dfrac{\Gamma, \theta \mid \phi, \quad \Delta, \theta \mid \neg\phi}{\Gamma \cup \Delta \mid \neg\theta}$	
Not Not Out (NNO)	$\dfrac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$	
Monotonicity (MON)	$\dfrac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}$	
All In ($\forall$I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \forall w_j\, \theta(w_j/x_i)}$	where x_i does not occur in any formula in Γ and w_j does not occur in θ
All Out ($\forall$O)	$\dfrac{\Gamma \mid \forall w_j\, \theta(w_j, \vec{x})}{\Gamma \mid \theta(t(\vec{x}), \vec{x})}$	for $t(\vec{x}) \in TL$
Exists In ($\exists$I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \exists\, w_j\, \theta'}$	where θ' is the result of replacing any number of occurences of the term $t(\vec{x})$ in θ by w_j and w_j does not occur in θ.
Exists Out ($\exists$O)	$\dfrac{\Gamma,\, \phi \mid \theta}{\Gamma, \exists w_j\, \phi(w_j/x_i) \mid \theta}$	where x_i does not occur in θ nor any formula in Γ and w_j does not occur in ϕ.
REF	$\Gamma \mid \theta$ whenever $\theta \in \Gamma$.	

The Equality Axioms, EqL

Eq1 $\forall w_1\, w_1 = w_1$

Eq2 $\forall w_1, w_2\, (w_1 = w_2 \to w_2 = w_1)$

Eq3 $\forall w_1, w_2, w_3\, ((w_1 = w_2 \wedge w_2 = w_3) \to w_1 = w_3)$

Eq4

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (R(w_1, w_2, \ldots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$$

for R an r-ary relation symbol of L.

Eq5

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to f(w_1, w_2, \ldots, w_r) = f(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$$

for f an r-ary function symbol of L.

Eq6

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to t(w_1, w_2, \ldots, w_r) = t(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$$

for $t(x_1, x_2, \ldots, x_r) \in TL$.

Eq7

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (\theta(w_1, w_2, \ldots, w_r) \leftrightarrow \theta(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$$

for $\theta(x_1, x_2, \ldots, x_r) \in FL$.

END OF EXAMINATION PAPER