MATH33001/43001/63001 First Coursework, Solutions and Feedback, 2013-2014

The Problems

1. Let the language L have a single binary relation symbol R. Which of the following are formulae of L? You should very briefly justify your answers.

(b) $\neg(\exists w_1\, R(x_1, x_1))$

(c) $(R(x_1, x_2) \wedge \exists w_1\, R(w_1, w_1))$

Let M be the structure for L with $|M|$ the set of all subsets of $\mathbb{N}$ and

$$R^M = \{\, \langle s, t\rangle \in |M|^2 \mid s \cap t = \emptyset \,\}.$$

Which of the following are true in M?

(e) $\forall w_1 \forall w_2\, (\neg R(w_1, w_2) \to \neg R(w_1, w_1))$

(f) $\forall w_1 \exists w_2\, (R(w_1, w_2) \wedge \forall w_3\, (\neg R(w_3, w_3) \to \neg(R(w_1, w_3) \wedge R(w_2, w_3))))$

Write down formulae $\theta_1(x_1), \theta_2(x_1, x_2), \theta_3(x_1)$ of L such that for $s, t \in |M|$,

$$M \models \theta_1(s) \iff s = \emptyset,$$

$$M \models \theta_2(s, t) \iff s \subseteq t,$$

$$M \models \theta_3(s) \iff s \text{ has at least two elements.}$$

Let K be the structure for L with $|K| = \mathbb{N}$ and

$$R^K = \{\, \langle n, m\rangle \in |K|^2 \mid n \leq 2m \,\}.$$

Write down a sentence η such that $M \models \eta$ and $K \models \neg\eta$.

2.[1] Write down a sentence in Prenex Normal Form logically equivalent to

$$(\forall w_1\, S(w_1) \to \neg(\forall w_2\, P(w_2) \vee \exists w_1\, Q(w_1)))$$

where P, Q, S are unary relation symbols.

[1]Only for level 4& 6 students.

The Solutions

1. (a) Not a formula of L since it contains different numbers of left and right parentheses – and we can prove by induction on the length of formulae that they always have the same number of right and left parentheses.

(b) Not a formula of L since we can prove by induction on $|\theta|$ for $\theta \in FL$ that the number of left parentheses, i.e. (, occurring in θ equals the number of relation symbols plus the number of binary connectives occurring in θ whereas for $\neg(\exists w_1 R(x_1, x_1))$ these are 2 and 1 respectively. [The key point you should be aware of here is that when we introduce negation, $\neg$, we do not introduce any parentheses.]

(c) This is a formula of L since $R(x_1, x_2), R(x_3, x_3) \in FL$ by L1, so from the latter, $\exists w_1 \, R(w_1, w_1) \in FL$ by L3 and then by L2, $(R(x_1, x_2) \wedge \exists w_1 \, R(w_1, w_1)) \in FL$.

(d) It is enough to simply say that this is true but for the record:

$$
\begin{aligned}
& M \models \exists w_1 \forall w_2 \, R(w_1, w_2) \\
& \qquad \iff \text{for some } s \in |M|, \text{ for all } t \in |M|, \; M \models R(s,t) \\
& \qquad \iff \text{for some } s \in |M|, \text{ for all } t \in |M|, \langle s, t \rangle \in R^M \\
& \qquad \iff \text{for some } s \in |M|, \text{ for all } t \in |M|, s \cap t = \emptyset
\end{aligned}
$$

— which is true when we take $s = \emptyset$, since $\emptyset \cap t = \emptyset$ for any set t.

(e) True. Again, for the record, let $s, t \in |M|$ and suppose that $M \models \neg R(s,t)$. Then not $M \models R(s,t)$, i.e. not $s \cap t = \emptyset$, so s cannot be empty and $s \cap s = s \neq \emptyset$. Hence $M \models \neg R(s,s)$. This shows that

$$M \models \neg R(s,t) \;\Rightarrow\; M \models \neg R(s,s),$$

so

$$M \models (\neg R(s,t) \to \neg R(s,s)),$$

and $M \models \forall w_1 \forall w_2 \, (\neg R(w_1, w_2) \to \neg R(w_1, w_1))$ since s, t were arbitrary elements of $|M|$.

(f) True. Again just saying this is enough to get the marks but for the record let $s \in |M|$ and let $t = \mathbb{N} - s$. Now let r be an element of $|M|$ and suppose that $M \models \neg R(r,r)$, so $r \cap r \neq \emptyset$, i.e. $r \neq \emptyset$. In this case we cannot have $r \cap s = r \cap t = \emptyset$ since otherwise

$$\emptyset \neq r = r \cap \mathbb{N} = r \cap (s \cup (\mathbb{N} - s)) = (r \cap s) \cup (r \cap (\mathbb{N} - s)) = \emptyset \cup \emptyset = \emptyset.$$

So not $M \models R(s,r) \wedge R(t,r)$, equivalently $M \models \neg(R(s,r) \wedge R(t,r))$. Summing up we have shown that for every $s \in |M|$ there is a $t \in |M|$ such that for all $r \in |M|$, if $M \models \neg R(r,r)$ then $M \models \neg(R(s,r) \wedge R(t,r))$, equivalently

$$M \models \forall w_1 \exists w_2 \forall w_3 \, (\neg R(w_3, w_3) \to \neg(R(w_1, w_3) \wedge R(w_2, w_3))).$$

$$\begin{aligned}
&\theta_1(x_1): \quad R(x_1, x_1) \\
&\theta_2(x_1, x_2): \quad \forall w_1 \, (R(w_1, x_2) \to R(w_1, x_1)) \\
&\theta_3(x_1): \quad \exists w_1 \exists w_2 \, (R(w_1, w_2) \wedge (\neg R(w_1, x_1) \wedge \neg R(w_2, x_1))).
\end{aligned}$$

[There are many other possibilities for $\theta_1(x_1), \theta_2(x_1, x_2), \theta_3(x_1, x_2, x_3)$.]

A suitable sentence η is

$$\forall w_1 \forall w_2 \, (R(w_1, w_2) \to R(w_2, w_1)) \qquad \star$$

Again no need to say more than this but in case you don't see why this works, for any $s, t \in |M|$, if $s \cap t = \emptyset$ then $t \cap s = \emptyset$. In other words R^M is symmetric, equivalently $\star$ holds in M. However R^K is not symmetric, for example we have $1 \leq 2 \times 3$ (i.e. $K \models R(1,3)$) but we do not have $3 \leq 2 \times 1$ (i.e we do not have $K \models R(3,1)$) so $\star$ fails in K.

2. A sentence in Prenex Normal Form (there are many of course) logically equivalent to

$$(\forall w_2 \, S(w_2) \to \neg(\forall w_1 \, P(w_1) \vee \exists w_1 \, Q(w_1)))$$

is

$$\exists w_2 \exists w_1 \forall w_3 \, (S(w_1) \to (\neg P(w_2) \wedge \neg Q(w_3))).$$

Just writing this down is enough for the marks, but if you'd like to see the working notice that

$$\begin{aligned}
&(\forall w_2 \, S(w_2) \to \neg(\forall w_1 \, P(w_1) \vee \exists w_1 \, Q(w_1))) \\
&\qquad\qquad \equiv \exists w_2 \, (S(w_2) \to \neg(\forall w_1 \, P(w_1) \vee \exists w_1 \, Q(w_1))) \qquad (1)
\end{aligned}$$

by the Useful Equivalents, UEs for short. Again by the UEs, the transitivity of $\equiv$ and Lemma 1,

$$\begin{aligned}
\neg(\forall w_1\, P(w_1) \vee \exists w_1\, Q(w_1)) &\equiv (\neg\forall w_1\, P(w_1) \wedge \neg\exists w_1\, Q(w_1)) \\
&\equiv (\exists w_1\, \neg P(w_1) \wedge \forall w_1\, \neg Q(w_1)) \\
&\equiv (\exists w_1\, \neg P(w_1) \wedge \forall w_3\, \neg Q(w_3)) \\
&\equiv \exists w_1\, (\neg P(w_1) \wedge \forall w_3\, \neg Q(w_3)) \\
&\equiv \exists w_1 \forall w_3\, (\neg P(w_1) \wedge \neg Q(w_3)). \qquad (2)
\end{aligned}$$

From (2), $\quad (S(x_2) \to \neg(\forall w_1\, P(w_1) \vee \exists w_1\, Q(w_1)))$

$$\begin{aligned}
&\equiv (S(x_2) \to \exists w_1 \forall w_3\, (\neg P(w_1) \wedge \neg Q(w_3))) \\
&\equiv \exists w_1\, (S(x_2) \to \forall w_3\, (\neg P(w_1) \wedge \neg Q(w_3))), \\
&\quad \text{by the UEs and Lemma 1,} \\
&\equiv \exists w_1 \forall w_3\, (S(x_2) \to (\neg P(w_1) \wedge \neg Q(w_3))). \qquad (3)
\end{aligned}$$

From (1) we have the equivalence of

$$(\forall w_2\, S(w_2) \to \neg(\forall w_1\, P(w_1) \vee \exists w_1\, Q(w_1)))$$

with

$$\exists w_2\, (S(w_2) \to \neg(\forall w_1\, P(w_1) \vee \exists w_1\, Q(w_1)))$$

with which in turn by (3) and Lemma 1, is equivalent to the Prenex Normal Form formula

$$\exists w_2 \exists w_1 \forall w_3\, (S(w_2) \to (\neg P(w_1) \wedge \neg Q(w_3))).$$

Transitivity and (1) now gives the equivalence of this with the original formula

$$(\forall w_2\, S(w_2) \to \neg(\forall w_1\, P(w_1) \vee \exists w_1\, Q(w_1))).$$

The Feedback

If you had any parts wrong and cannot see why from the marker's comments[2] on your script then check out the model answer above. You are quite likely to get questions like these in the exam so make sure you don't repeat the same mistakes there – when you won't have a lot of time.

In addition to the specific comments on the scripts here are some general observations:

[2]This year I was not the marker for this Coursework.

1. Most students wrote much more than was needed. If I ask for a formula, as in the last four parts, just give a formula don't give an argument why you think your formula does the job. Similar if I ask whether or not $M \models \theta$ just say one of 'true'/'false'. Indeed one danger in giving a long explanation of your answer is that the answer might be right but the explanation patently wrong! (Actually in such cases I adopted the position that within reason one doesn't penalize students for saying something silly, though sometimes it is difficult to swallow.) If I want some explanation I'll ask for it.

2. When a formula was asked for the answer given frequently wasn't a formula. A particularly common mistake was to omit brackets, or insert too many brackets. This was particularly the case when a conjunction was written as $\theta \wedge \phi \wedge \psi$ instead of $(\theta \wedge \phi) \wedge \psi$ (or $\theta \wedge (\phi \wedge \psi)$) for $\theta, \phi, \psi \in FL$. Generally such errors were not penalized when it was clear how the 'formula' could be repaired. In the case that the 'formula' contained symbols which weren't in the language, such as 1,$\emptyset$, $\cap$, then no marks were given though the use of s, t in place of x_1, x_2 was pointed out as incorrect (make sure you see why) but not otherwise penalized.

3. Some of you still start your solution by re-writing the question. This is risky, the number of times students do this in an exam and make a copying mistake you wouldn't believe.

4. The expression $\neg(\exists w_1 \, R(x_1, x_1))$ in 1(b) isn't a formula, but not, as many students asserted, because w_1 does not occur in $R(x_1, x_1)$, L3 does not stipulate that it has to. The reason it is not a formula is because it contains to many left (or right) parentheses.

5. Some students seemed unaware that $\theta \to \phi$ will be true if θ is false.

6. For Question 2 on the level 4&6 version most students got a correct PNF, but those who didn't seemed to have misapplied or misinterpreted some of the UEs, particularly the one about taking the quantifier out from the LHS of an implication to the whole of the implication.