Two hours

THE UNIVERSITY OF MANCHESTER

PREDICATE LOGIC

23 January 2014

9.45 – 11.45

Answer ALL FOUR questions in Section A (56 marks in all).
Answer TWO of the THREE questions in Section B (24 marks in total).

If more than TWO questions from Section B are attempted,
then credit will be given for the FIRST TWO answers.

A list of axioms and rules of proof is appended to this examination paper

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Electronic calculators are not permitted

SECTION A

Answer ALL FOUR questions

A1. Do you think the following argument is valid in the sense that the conclusion ‘follows’ from the premises? You should briefly explain your answer.

Some numbers are divisible by 3
Some numbers are divisible by 5

$\therefore$ *Some numbers are divisible by both 3 and 5*

[4 marks]

A2. Let the language L have a binary relation symbol R and a binary function symbol f. Which of the following are terms of L? You should justify your answers.

(i) $f(x_1, f(x_1, x_1))$

(ii) $f((f(x_1, x_2), x_1)))$

Which of the following are formulae of L? You should briefly justify your answers.

(iii) $\forall w_1\, (R(x_1, x_1) \vee R(x_1, x_1))$

(iv) $\forall x_1\, (R(x_1, x_1) \vee \neg R(x_1, x_1))$

Let M be the structure for L with $|M| = \{2, 3, 4, \ldots\}$, $f^M(n, m) = n \times m$, and

$$R^M = \{\, \langle n, m\rangle \in |M|^2 \mid n < m \,\}.$$

Which of the following sentences of L are true in M?

(v) $\forall w_1 \exists w_2\, R(w_2, w_1)$,

(vi) $\forall w_1 \forall w_2\, (\exists w_3\, R(f(w_1, w_3), f(w_2, w_3)) \to R(w_1, w_2))$,

(vii) $\exists w_1 \forall w_2\, (R(w_2, w_1) \to R(f(w_2, w_2), w_1))$.

Find formulae $\theta_1(x_1, x_2)$, $\theta_2(x_1)$, $\theta_3(x_1, x_2)$, $\theta_4(x_1)$ of L such that for $n \in |M|$,

$$\begin{aligned} M \models \theta_1(n, m) &\iff n = m, \\ M \models \theta_2(n) &\iff n = 2, \\ M \models \theta_3(n, m) &\iff n^2 \le m, \\ M \models \theta_4(n) &\iff n = 3. \end{aligned}$$

Let K be the structure for L with $|K| = \mathbb{N} = \{\, 1, 2, 3, \ldots\}$, $f^K(n, m) = n \times m$,

$$R^K = \{\, \langle n, m\rangle \in |K|^2 \mid n < m \,\}.$$

Find a sentence ϕ of L such that $K \vDash \phi$ and $M \nvDash \phi$.

[28 marks]

A3. Define what is meant by a *formal proof*. Give a formal proof of

$$\forall w_1\,(P(w_1) \to Q(w_1)), \exists w_1\, P(w_1) \vdash \exists w_1\, Q(w_1)$$

where P, Q are unary relation symbols.

[11 marks]

A4. State the *Completeness Theorem*. Using this theorem or otherwise show that

(a) $\forall w_1\, P(g(w_1)) \nvDash \forall w_1\, P(w_1)$

(b) $\forall w_1\,(P(w_1) \to \neg P(g(w_1))) \vdash \exists w_1\, \neg P(w_1)$

where P is a unary relation symbol and g a unary function symbol.

Does

$$P(x_1) \to \neg P(g(x_1)), P(x_2) \vdash \neg P(g(x_2)) \quad ?$$

Justify your answer.

[13 marks]

SECTION B

Answer TWO of the THREE questions

If more than TWO questions are attempted then credit will be given for the FIRST TWO answers.

B5. Let L be a relational language and for $\theta \in FL$ let θ^* be the expression resulting from removing every occurrence of $\neg$ in θ. Give the main points of a proof that for $\theta \in FL$, $\theta^* \in FL$.

Does this conclusion still hold if instead every occurrence of $\wedge$ in θ is removed? Briefly justify your answer.

[12 marks]

B6. Give a formal proof of

$$\forall w_1 P(w_1) \vee \forall w_1 \neg P(w_1) \vdash \neg \exists w_1 \exists w_2 \,(P(w_1) \wedge \neg P(w_2))$$

where P is a unary relation symbol.

[12 marks]

B7. State the *Compactness Theorem for Normal Structures.*

Let L be a language with equality. Show that there can be no sentence θ of L such that for M a normal structure for L,

$$M \models \theta \iff |M| \text{ is finite.}$$

[12 marks]

The Rules of Proof and Axiom for the Predicate Calculus

And In (AND)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \phi}{\Gamma \cup \Delta \mid \theta \wedge \phi}$	
And Out (AO)	$\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta}$ $\qquad$ $\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$	
Or In (ORR)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi}$ $\qquad$ $\dfrac{\Gamma \mid \theta}{\Gamma \mid \phi \vee \theta}$	
Disjunction (DIS)	$\dfrac{\Gamma, \theta \mid \psi, \quad \Delta, \phi \mid \psi}{\Gamma \cup \Delta, \theta \vee \phi \mid \psi}$	
Implies In (IMR)	$\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$	
Modus Ponens (MP)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \theta \to \phi}{\Gamma \cup \Delta \mid \phi}$	
Not In (NIN)	$\dfrac{\Gamma, \theta \mid \phi, \quad \Delta, \theta \mid \neg\phi}{\Gamma \cup \Delta \mid \neg\theta}$	
Not Not Out (NNO)	$\dfrac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$	
Monotonicity (MON)	$\dfrac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}$	
All In (∀I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \forall w_j\, \theta(w_j/x_i)}$	where x_i does not occur in any formula in Γ and w_j does not occur in θ
All Out (∀O)	$\dfrac{\Gamma \mid \forall w_j\, \theta(w_j, \vec{x})}{\Gamma \mid \theta(t(\vec{x}), \vec{x})}$	for $t(\vec{x}) \in TL$
Exists In (∃I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \exists\, w_j\, \theta'}$	where θ' is the result of replacing any number of occurences of the term $t(\vec{x})$ in θ by w_j and w_j does not occur in θ.
Exists Out (∃O)	$\dfrac{\Gamma,\, \phi \mid \theta}{\Gamma, \exists w_j \phi(w_j/x_i) \mid \theta}$	where x_i does not occur in θ nor any formula in Γ and w_j does not occur in ϕ.
REF	$\Gamma \mid \theta$ whenever $\theta \in \Gamma$.	

The Equality Axioms, EqL

Eq1 $\forall w_1 \, w_1 = w_1$

Eq2 $\forall w_1, w_2 \, (w_1 = w_2 \to w_2 = w_1)$

Eq3 $\forall w_1, w_2, w_3 \, ((w_1 = w_2 \wedge w_2 = w_3) \to w_1 = w_3)$

Eq4

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (R(w_1, w_2, \ldots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$$

for R an r-ary relation symbol of L.

Eq5

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{n+i} \right) \to f(w_1, w_2, \ldots, w_r) = f(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$$

for f an r-ary function symbol of L.

Eq6

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to t(w_1, w_2, \ldots, w_r) = t(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$$

for $t(x_1, x_2, \ldots, x_r) \in TL$.

Eq7

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (\theta(w_1, w_2, \ldots, w_r) \leftrightarrow \theta(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$$

for $\theta(x_1, x_2, \ldots, x_r) \in FL$.

END OF EXAMINATION PAPER