

Two hours

**THE UNIVERSITY OF MANCHESTER**

PREDICATE LOGIC

17th January 2012

9.45 – 11.45

Answer **ALL** four questions in Section A (56 marks in all) and  
**TWO** questions in Section B (24 marks in all).

If you answer more than two questions from Section B only the first two  
appearing in your answer book will be marked.

**A list of axioms and rules of proof is appended to this examination paper**

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**The use of calculators is not permitted**

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**SECTION A**Answer **ALL** four questions

**A1.** Is the following argument is valid in the sense that the conclusion ‘follows’ from the premises? You should briefly explain your answer.

There is a frog who plays the piano and sings

$\therefore$  There is a frog who plays the piano

[5 marks]

**A2.** Let the language  $L$  have a ternary (i.e. 3-ary) relation symbol  $R$  and a unary function symbol  $f$ . Which of the following are terms of  $L$ ? You should briefly justify your answers.

(i)  $f(f(w_2))$ (ii)  $f(f(f(f(x_1))))$ 

Which of the following are formulae of  $L$ ? You should briefly justify your answers.

(iii)  $\forall w_1 \neg R(w_1, x_1, w_1)$ (iv)  $\forall w_1 R(w_1, x_1, w_1) \vee R(w_1, w_1, x_1)$ 

Let  $M$  be the structure for  $L$  with  $|M| = \mathbb{N} = \{0, 1, 2, 3, \dots\}$ ,  $f^M(n) = n + 1$ ,

$$R^M = \{ \langle n, m, k \rangle \in \mathbb{N}^3 \mid n < m < k \}.$$

Which of the following sentences of  $L$  are true in  $M$ ?

(v)  $\forall w_1 R(w_1, f(w_1), f(f(w_1)))$ (vi)  $\exists w_1 \exists w_2 R(w_1, w_2, f(w_1))$ (vii)  $\forall w_1 \forall w_2 \exists w_3 (R(w_1, w_3, f(f(w_2))) \vee R(w_2, w_3, f(f(w_1))))$ 

Find formulae  $\theta_1(x_1, x_2)$ ,  $\theta_2(x_1, x_2)$ ,  $\theta_3(x_1, x_2)$ ,  $\theta_4(x_1)$  of  $L$  such that for  $n, m \in |M|$ ,

$$M \models \theta_1(n, m) \iff n < m$$

$$M \models \theta_2(n, m) \iff n = m$$

$$M \models \theta_3(n, m) \iff |n - m| \geq 2$$

$$M \models \theta_4(n) \iff n = 0$$

Let  $K$  be the structure for  $L$  with  $|K| = |M| = \mathbb{N} = \{0, 1, 2, 3, \dots\}$ ,  $f^K(n) = n^2$ ,

$$R^K = R^M = \{ \langle n, m, k \rangle \in \mathbb{N}^3 \mid n < m < k \}.$$

Find a sentence  $\phi$  of  $L$  such that  $K \models \phi$  and  $M \not\models \phi$ .

[27 marks]

**A3.** Define what is meant by a *formal proof*? Give a formal proof of

$$\forall w_1 P(w_1) \vdash \forall w_1 Q(w_1) \rightarrow \forall w_1 (P(w_1) \wedge Q(w_1))$$

where  $P, Q$  are unary relation symbols.

[12 marks]

**A4.** State the Completeness Theorem. Using this theorem or otherwise show that

(a)  $\exists w_1 P(w_1) \not\vdash \exists w_1 Q(w_1) \rightarrow \exists w_1 (P(w_1) \wedge Q(w_1))$

(b)  $\forall w_1 (P(w_1) \vee Q(w_1)) \vdash \forall w_1 P(w_1) \vee \exists w_1 Q(w_1)$

where  $P, Q$  are unary relation symbols.

[12 marks]

**SECTION B**Answer **TWO** of the three questions

**B5.** Let  $L$  be the language with a single binary relation symbol  $R$ . Show that no two of the following sentences of  $L$  logically imply the third:

- (i)  $\forall w_1 \exists w_2 \neg R(w_1, w_2)$
- (ii)  $\exists w_1 \forall w_2 R(w_2, w_1)$
- (iii)  $\forall w_1 \forall w_2 (R(w_1, w_2) \vee R(w_2, w_1))$

[12 marks]

**B6.** Give a formal proof of

$$\forall w_1 P(w_1) \vdash \neg \exists w_1 (\neg P(w_1) \wedge Q(w_1))$$

where  $P, Q$  are unary relation symbols.

[12 marks]

**B7.** State the Compactness Theorem for Normal Structures.

Let  $L$  be a language with equality. Show that there can be no sentence  $\theta$  of  $L$  such that for  $M$  a normal structure for  $L$ ,

$$M \models \theta \iff |M| \text{ is finite.}$$

[12 marks]

## The Rules of Proof and Axiom for the Predicate Calculus

And In (AND)	$\frac{\Gamma   \theta, \Delta   \phi}{\Gamma \cup \Delta   \theta \wedge \phi}$	
And Out (AO)	$\frac{\Gamma   \theta \wedge \phi}{\Gamma   \theta} \quad \frac{\Gamma   \theta \wedge \phi}{\Gamma   \phi}$	
Or In (ORR)	$\frac{\Gamma   \theta}{\Gamma   \theta \vee \phi} \quad \frac{\Gamma   \theta}{\Gamma   \phi \vee \theta}$	
Disjunction (DIS)	$\frac{\Gamma, \theta   \psi, \Delta, \phi   \psi}{\Gamma \cup \Delta, \theta \vee \phi   \psi}$	
Implies In (IMR)	$\frac{\Gamma, \theta   \phi}{\Gamma   \theta \rightarrow \phi}$	
Modus Ponens (MP)	$\frac{\Gamma   \theta, \Delta   \theta \rightarrow \phi}{\Gamma \cup \Delta   \phi}$	
Not In (NIN)	$\frac{\Gamma, \theta   \phi, \Delta, \theta   \neg \phi}{\Gamma \cup \Delta   \neg \theta}$	
Not Not Out (NNO)	$\frac{\Gamma   \neg \neg \theta}{\Gamma   \theta}$	
Monotonicity (MON)	$\frac{\Gamma   \theta}{\Gamma \cup \Delta   \theta}$	
All In ( $\forall$ I)	$\frac{\Gamma   \theta}{\Gamma   \forall w_j \theta(w_j/x_i)}$	where $x_i$ does not occur in any formula in $\Gamma$ and $w_j$ does not occur in $\theta$
All Out ( $\forall$ O)	$\frac{\Gamma   \forall w_j \theta(w_j, \vec{x})}{\Gamma   \theta(t(\vec{x}), \vec{x})}$	for $t(\vec{x}) \in TL$
Exists In ( $\exists$ I)	$\frac{\Gamma   \theta}{\Gamma   \exists w_j \theta'}$	where $\theta'$ is the result of replacing any number of occurrences of the term $t(\vec{x})$ in $\theta$ by $w_j$ and $w_j$ does not occur in $\theta$ .
Exists Out ( $\exists$ O)	$\frac{\Gamma, \phi   \theta}{\Gamma, \exists w_j \phi(w_j/x_i)   \theta}$	where $x_i$ does not occur in $\theta$ nor any formula in $\Gamma$ and $w_j$ does not occur in $\phi$ .
REF	$\Gamma   \theta$ whenever $\theta \in \Gamma$ .	

## The Equality Axioms, $Eq$

**Eq1**  $\forall w_1 w_1 = w_1$

**Eq2**  $\forall w_1, w_2 (w_1 = w_2 \rightarrow w_2 = w_1)$

**Eq3**  $\forall w_1, w_2, w_3 ((w_1 = w_2 \wedge w_2 = w_3) \rightarrow w_1 = w_3)$

**Eq4**

$$\forall w_1, \dots, w_{2r} \left( \left( \bigwedge_{i=1}^r w_i = w_{r+i} \right) \rightarrow (R(w_1, w_2, \dots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \dots, w_{2r})) \right)$$

for  $R$  an  $r$ -ary relation symbol of  $L$ .

**Eq5**

$$\forall w_1, \dots, w_{2r} \left( \left( \bigwedge_{i=1}^r w_i = w_{r+i} \right) \rightarrow f(w_1, w_2, \dots, w_r) = f(w_{r+1}, w_{r+2}, \dots, w_{2r}) \right)$$

for  $f$  an  $r$ -ary function symbol of  $L$ .

**Eq6**

$$\forall w_1, \dots, w_{2r} \left( \left( \bigwedge_{i=1}^r w_i = w_{r+i} \right) \rightarrow t(w_1, w_2, \dots, w_r) = t(w_{r+1}, w_{r+2}, \dots, w_{2r}) \right)$$

for  $t(x_1, x_2, \dots, x_r) \in TL$ .

**Eq7**

$$\forall w_1, \dots, w_{2r} \left( \left( \bigwedge_{i=1}^r w_i = w_{r+i} \right) \rightarrow (\theta(w_1, w_2, \dots, w_r) \leftrightarrow \theta(w_{r+1}, w_{r+2}, \dots, w_{2r})) \right)$$

for  $\theta(x_1, x_2, \dots, x_r) \in FL$ .