Two hours

UNIVERSITY OF MANCHESTER

PREDICATE LOGIC

17th January 2011

9.45 – 11.45

Answer **ALL** questions in Section A and **TWO** questions in Section B.

A list of axioms and rules of proof is appended to this examination paper

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Calculators may be used but only if they cannot store text.

SECTION A

Answer **ALL** four questions

A1. Do you think the following argument is valid in the sense that the conclusion 'follows' from the premises? You should briefly explain your answer.

Everyone who loves football loves Spain
Everyone loves Spain

$\therefore$ *Everyone loves football*

[5 marks]

A2. Let the language L have a binary relation symbol R and binary function symbol f. Which of the following are terms of L? You should justify your answers.

(i) $f(x_1, f(x_1, x_2))$

(ii) $f((f(x_1, x_2), x_1)$

Which of the following are formulae of L? You should justify your answers.

(iii) $\forall w_1 \neg R(w_1, x_1)$

(iv) $\forall w_1 \neg R(w_2, x_1)$

Let M be the structure for L with $|M| = \mathbb{N}^+ = \{1, 2, 3, \ldots\}$, $f^M(n, m) = nm$,

$$R^M = \{ \langle n, m \rangle \in |M|^2 \mid n < m \}.$$

Which of the following sentences of L are true in M?

(v) $\forall w_1 \forall w_2 \, (R(w_1, w_2) \to R(w_2, w_1))$

(vi) $\exists w_1 \forall w_2 \neg R(w_2, f(w_1, w_2))$

(vii) $\forall w_1 \, (R(w_1, f(w_1, w_1)) \to \forall w_2 \, R(w_2, f(w_1, w_2)))$

Find formulae $\theta_1(x_1, x_2)$, $\theta_2(x_1, x_2)$, $\theta_3(x_1, x_2)$, $\theta_4(x_1, x_2)$ of L such that for $n, m \in |M|$,

$$\begin{aligned} M \models \theta_1(n, m) &\iff n^2 < m \\ M \models \theta_2(n, m) &\iff n = m \\ M \models \theta_3(n, m) &\iff n + 1 = m \\ M \models \theta_4(n, m) &\iff n \text{ divides } m \end{aligned}$$

Let K be the structure for L with $|K| = \mathbb{N} = \{ 0, 1, 2, 3, \ldots \}$, $f^K(n, m) = nm$,

$$R^K = \{ \langle n, m \rangle \in |K|^2 \mid n < m \}.$$

Find a sentence ϕ of L such that $M \models \phi$ and $K \not\models \phi$.

[27 marks]

A3. Define what is meant by a *formal proof*? Give a formal proof of

$$\forall w_1 \, (P(w_1) \to Q(w_1)) \vdash \forall w_1 \, P(w_1) \to \forall w_1 Q(w_1)$$

where P, Q are unary relation symbols.

[12 marks]

A4. State the Completeness Theorem. Using this theorem or otherwise show that

(a) $\forall w_1 \, P(w_1) \to \forall w_1 \, Q(w_1) \nvdash \forall w_1 \, (P(w_1) \to Q(w_1))$

(b) $\forall w_1 \forall w_2 \, (P(w_1) \vee Q(w_2)) \vdash \forall w_1 \, P(w_1) \vee \exists w_2 \, Q(w_2)$

where P Q are unary relation symbols.

[12 marks]

SECTION B

Answer **TWO** of the three questions

B5. Let L be the language with a single binary relation symbol R. Show that no two of the following sentences of L logically imply the third:

(i) $\forall w_1 \exists w_2\, R(w_1, w_2)$

(ii) $\exists w_1 \forall w_2\, \neg R(w_2, w_1)$

(iii) $\forall w_1 \forall w_2 (R(w_1, w_2) \to \exists w_3\, (R(w_1, w_3) \wedge R(w_3, w_2)))$

[12 marks]

B6. Give a formal proof of

$$\exists w_1\, (P(w_1) \vee Q(w_1)),\ \forall w_1 \neg P(w_1) \vdash \exists w_1\, Q(w_1)$$

where P, Q are unary relation symbols.

[12 marks]

B7. State the Compactness Theorem for Normal Structures.

Let L be a language with equality. Show that there can be no sentence θ of L such that for M a normal structure for L,

$$M \models \theta \iff |M| \text{ is finite.}$$

[12 marks]

The Rules of Proof and Axiom for the Predicate Calculus

And In (AND)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \phi}{\Gamma \cup \Delta \mid \theta \wedge \phi}$		
And Out (AO)	$\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta}$	$\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$	
Or In (ORR)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi}$	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \phi \vee \theta}$	
Disjunction (DIS)	$\dfrac{\Gamma, \theta \mid \psi, \quad \Delta, \phi \mid \psi}{\Gamma \cup \Delta, \theta \vee \phi \mid \psi}$		
Implies In (IMR)	$\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$		
Modus Ponens (MP)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \theta \to \phi}{\Gamma \cup \Delta \mid \phi}$		
Not In (NIN)	$\dfrac{\Gamma, \theta \mid \phi, \quad \Delta, \theta \mid \neg\phi}{\Gamma \cup \Delta \mid \neg\theta}$		
Not Not Out (NNO)	$\dfrac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$		
Monotonicity (MON)	$\dfrac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}$		
All In ($\forall$I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \forall w_j\, \theta(w_j/x_i)}$		where x_i does not occur in any formula in Γ and w_j does not occur in θ
All Out ($\forall$O)	$\dfrac{\Gamma \mid \forall w_j\, \theta(w_j, \vec{x})}{\Gamma \mid \theta(t(\vec{x}), \vec{x})}$		for $t(\vec{x}) \in TL$
Exists In ($\exists$I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \exists\, w_j\, \theta'}$		where θ' is the result of replacing any number of occurences of the term $t(\vec{x})$ in θ by w_j and w_j does not occur in θ.
Exists Out ($\exists$O)	$\dfrac{\Gamma,\ \phi \mid \theta}{\Gamma, \exists w_j \phi(w_j/x_i) \mid \theta}$		where x_i does not occur in θ nor any formula in Γ and w_j does not occur in ϕ.
REF	$\Gamma \mid \theta$ whenever $\theta \in \Gamma$.		

The Equality Axioms, Eq

Eq1 $\forall w_1\, w_1 = w_1$

Eq2 $\forall w_1, w_2\, (w_1 = w_2 \to w_2 = w_1)$

Eq3 $\forall w_1, w_2, w_3\, ((w_1 = w_2 \wedge w_2 = w_3) \to w_1 = w_3)$

Eq4

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (R(w_1, w_2, \ldots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$$

for R an r-ary relation symbol of L.

Eq5

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{n+i} \right) \to f(w_1, w_2, \ldots, w_r) = f(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$$

for f an r-ary function symbol of L.

Eq6

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to t(w_1, w_2, \ldots, w_r) = t(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$$

for $t(x_1, x_2, \ldots, x_r) \in TL$.

Eq7

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (\theta(w_1, w_2, \ldots, w_r) \leftrightarrow \theta(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$$

for $\theta(x_1, x_2, \ldots, x_r) \in FL$.

END OF EXAMINATION PAPER