ADVANCED MATHEMATICAL METHODS

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1 Introduction

This course aims to introduce and discuss a number of commonly used mathematical methods techniques that graduate students will find useful in their research. In the subsequent lectures we will aim to cover the following topics:

- Advanced differential equations, series solution,classification of singularities. Properties near ordinary and regular singular points. Approximate behaviour near irregular singular points. Method of dominant balance. Airy, Gamma and Bessel functions.

- Asymptotic methods. Boundary layer theory. Regular and singular perturbation problems. Uniform approximations. Interior layes. LG approximation, WKBJ method.

- Generalised functions. Basic definitions and properties.

- Revision of basic complex analysis. Laurent expansions. Singularities. Cauchy's Theorem. Residue calculus. Plemelj formuale.

- Transform methods. Fourier transform. FT of generalised functions. Laplace Transform. Properties of Gamma function. Mellin Transform. Analytic continuation of Mellin transforms.

- Asymptotic expansion of integrals. Laplace's method. Watson's Lemma. Method of stationary phase. Method of steepest descent. Estimation using Mellin transform technique.

- Conformal mapping. Riemann-Hilbert problems.

Many of the above topics could easily be studied in detail over many lectures, but our motivation is to give a flavour of the particular topic rather than give an exhaustive treatment of the subject. There will be sufficient detail for the interested reader to follow up and investigate further if required. The course therefore proceeds at a fairly rapid pace and students are strongly advised to study the techniques, work through the examples covered and also attempt the set problems.

Recommended Texts It will be assumed that students have done basic courses on real and complex analysis. The following texts cover various topics discussed in the course, although no single book covers all the topics that we will be discussing.

1. C. M. Bender & S.A. Orszag, 'Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill.

2. N. Bleistein & R.A. Handelsman, 'Asymptotic Expansions of Integrals.'

3. F. W. J. Olver, ‘Introduction to Asymptotics and Special Functions’, Dover.

4. M. J. Ablowitz & A. S. Fokas ‘Complex variables, introduction and applications’, C.U.P.

5. M. J. Lighthill ‘Introduction to Fourier analysis and generalised functions.’, Dover.

2 Important definitions and preliminaries

In this section we will introduce some of the definitions and notation which will be used extensively in later parts of the course.

2.1 Ordering symbols, 'O and 'o' notation

Ordering symbols 'O' and 'o'

Definition of 'O': *Let $\phi(x), \psi(x)$ be real or complex valued functions. Let x_0 be a limit point of a set R not necessarily belonging to R. We write*

$$\psi = O(\phi) \quad in \quad R$$

if $\exists$ a constant A (independent of x) so that

$$|\psi| \leq A|\phi| \quad \forall x \in R.$$

Also $\psi = O(\phi)$ as $x \to x_0$ in some neighbourhood Δ, if $\exists A$ such that

$$|\psi| \leq A|\phi| \quad \forall x \in \Delta \cap R.$$

If $\phi \neq 0$ in R then $\psi = O(\phi)$ as $x \to x_0$ if $\frac{\psi}{\phi}$ is bounded in R as $x \to x_0$.

Examples

$$\sin x = O(x) \quad \text{as} \quad x \to 0.$$
$$\cos x = O(1) \quad \text{as} \quad x \to 0.$$

Definition of 'o': *We write $\psi = o(\phi)$ as $x \to x_0$ if for any given $\epsilon > 0 \quad \exists$ neighbourhood Δ_ϵ of x_0 such that*

$$|\psi| \leq A\epsilon|\phi| \quad \forall x \in \Delta_\epsilon \cap R.$$

Note that if $\phi \neq 0$ in R then $\psi = o(\phi)$ as $x \to x_0$ if $\frac{\psi}{\phi} \to 0$ as $x \to x_0$.

Sometimes $<<$ used in place of o notation.

Examples

$$\sin x = o(1) \quad \text{as} \quad x \to 0.$$

If the functions involved depend on parameters, in general the constants A, and neighbourhoods Δ, Δ_ϵ will depend on the parameters.

If however, $A, \Delta, \Delta_\epsilon$ are independent of the parameters, the order relation is said to hold uniformly in the parameters.

Examples

$$\sin(x+\epsilon) = O(1) \quad \text{uniformly as} \quad x \to 0.$$
$$\sqrt{x+\epsilon} - \sqrt{x} = O(\epsilon) \quad \text{nonuniformly as} \quad \epsilon \to 0.$$
$$\sin(x+\epsilon) = o(\epsilon^{-\frac{1}{2}}) \quad \text{uniformly as} \quad \epsilon \to 0.$$

Video clip of section on the O and o notation.

2.2 Asymptotic sequences

Asymptotic sequences are extremely useful and will be used throughout this course.

Definition *The sequence of functions* $\{\phi_n\}$ *is called an* **asymptotic sequence** *for* $x \to x_0$ *in* R *if for each* n, ϕ_n *is defined in* R *and*

$$\phi_{n+1} = o(\phi_n) \quad as \quad x \to x_0 \quad in \quad R.$$

If the sequence is infinite and $\phi_{n+1} = O(\phi_n)$ *uniformly in* n*, then the* $\{\phi_n\}$ *is said to be an asymptotic sequence uniformly in* n*. If the* ϕ_n *depend on parameters, and* $\phi_{n+1} = o(\phi_n)$ *in the parameters, the* $\{\phi_n\}$ *is an asymptotic sequence uniformly in the parameters.*

Example The following define asymptotic sequences

$$\{(x-x_0)^n\} \quad x \to x_0 \quad x \in C.$$

$$\{x^{-n}\}, \quad \text{as} \quad x \to \infty.$$
$$\{x^{-\lambda_n}\}, \quad \text{as} \quad x \to \infty,$$

where $\Re(\lambda_n) < \Re(\lambda_{n+1})$ for each n.

Definition *We say*

$$f(x) \sim g(x) \quad as \quad x \to x_0$$

if

$$\frac{f(x)}{g(x)} \to 1 \quad as \quad x \to x_0.$$

Observe that this implies

$$f(x) = (1 + o(1))g(x) \quad \text{as} \quad x \to x_0.$$

2.3 Asymptotic expansion

Definition *Let* $\{\phi_n\}$ *be an asymptotic sequence. The series*

$$\sum a_n \phi_n(x)$$

is said to be an **asymptotic expansion** *to* N *terms of* $f(x)$ *as* $x \to x_0$ *if*

$$f(x) - \sum_{n=1}^{N} a_n \phi_n(x) = O(\phi_{N+1}) \quad as \quad x \to x_0.$$

Sometimes this is written as

$$f(x) \sim \sum a_n \phi_n(x) \quad to \quad N \quad terms\ as \quad x \to x_0 \quad in \quad R.$$

If $N = \infty$ *then*

$$f(x) \sim \sum a_n \phi_n(x)$$

is called an asymptotic expansion. An asymptotic expansion involving certain parameters is said to hold uniformly in the parameters if

$$f - \sum_{n=1}^{N} a_n \phi_n(x) = O(\phi_{N+1})$$

uniformly in the parameters for each sufficiently large N, (not necessarily uniformly in N). If $\phi_n = x^{-\lambda_n}$ where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \dots$ then

$$\frac{\phi_{n+1}}{\phi_n} = \frac{x^{\lambda_n}}{x^{\lambda_{n+1}}} = \frac{1}{x^{\lambda_{n+1}-\lambda_n}} \to 0 \quad \text{as} \quad x \to \infty.$$

The above definitions stem from Poincaré (1886) studies. Poincaré (1886) introduced asymptotic power series as a means for making divergent series more useful. In Poincare's definition the point x_0 is infinity and the asymptotic sequence is z^{-n} where $z \to \infty$ in some section in the complex plane.

Poincaré power series expansions *A series* $\sum_{n=0}^{\infty} a_n z^{-n}$ *is called an asymptotic expansion of* $f(z)$ *in some sector* S, $\quad \alpha \leq \arg(z) \leq \beta$ *if for each* $N \geq 0$

$$f(z) = \sum_{n=0}^{N} a_n z^{-n} + O(z^{-(N+1)}), \quad z \to \infty.$$

Examples Consider

$$\sqrt{x+\epsilon} = \sqrt{x}(1+\frac{\epsilon}{x})^{\frac{1}{2}}.$$

This suggests

$$\sqrt{x+\epsilon} \sim \sqrt{x}\left[1+\frac{\epsilon}{2x}-\frac{\epsilon^2}{8x^2}+\dots\right].$$

Define

$$\phi_n(x,\epsilon) = \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!}\frac{\epsilon^n}{x^n},$$

with $x > 0$ and fixed n. Now

$$\frac{\phi_{n+1}}{\phi_n} = \frac{(\frac{1}{2}-n)}{(n+1)}\frac{\epsilon}{x} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Thus $\sum \phi_n$ is an asymptotic expansion.

Note that that the series

$$\sum_{n=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!}\frac{\epsilon^n}{x^n}$$

converges only for $|\epsilon| < |x|$. **Thus the series in an asymptotic expansion does not necessarily converge.**

Clip covering asymptotic expansions.

How do we know that $\sqrt{x+\epsilon} \sim x^{\frac{1}{2}} \sum \phi_n(x,\epsilon)$ above?

Theorem *The asymptotic expansion to a given number of terms of a given function is unique if the asymptotic sequence is given.*

Proof

If $f(x) \sim \sum a_n\phi_n(x)$ then

$$f(x) = \sum_{k=1}^{n} a_k\phi_k + R_n(x)$$

where $R_n(x) = o(\phi_n)$.

Hence

$$f(x) = \sum_{k=1}^{n-1} a_k\phi_k + a_n\phi_n + R_n(x)$$

Therefore

$$\left|\frac{f(x) - \sum_{k=1}^{n-1} a_k\phi_k}{\phi_n} - a_n\right| = \left|\frac{R_n}{\phi_n}\right| \to 0 \quad \text{as} \quad x \to x_0.$$

Hence a_n is given uniquely by

$$a_n = \lim_{x\to x_0}\left(\frac{f(x) - \sum_{k=1}^{n-1} a_k\phi_k(x)}{\phi_n(x)}\right) \tag{2.1}$$

Conversely, *suppose we have $N+1$ functions $f(x), \phi_1(x), \ldots \phi_N(x)$ defined in R. Then if (2.1) holds and $a_m \neq 0$ for $m = 1, 2, \ldots, N$ then $\{\phi_n\}$ is an asymptotic sequence for $x \to x_0$ and $\sum a_n\phi_n$ is an asymptotic expansion to N terms of $f(x)$ as $x \to x_0$.*

Proof: We have to show that $\phi_{n+1} = o(\phi_n)$ for $n = 1, 2, \ldots, N-1$. Now from (2.1)

$$f - \sum_{k=1}^{m} a_k\phi_k = o(\phi_m).$$

Replace m by $m+1$ and we have

$$f - \sum_{k=1}^{m} a_k\phi_k = a_{m+1}\phi_{m+1} + o(\phi_{m+1}).$$

$$= a_{m+1}\phi_{m+1} + o(1)\phi_{m+1},$$
$$= (a_{m+1} + o(1))\phi_{m+1}.$$

Hence

$$(a_{m+1} + o(1))\phi_{m+1} = o(\phi_m).$$

Thus if $a_{m+1} \neq 0$ then $q_{m+1} + o(1) \neq 0$ for some x in the neighbourhood of x_0 and dividing gives the result

$$\phi_{m+1} = o(\phi_m).$$

The same function may have different asymptotic expansions involving two different asymptotic sequences, or two different functions may have the same asymptotic expansion.

Examples

$$\frac{1}{x+1} = \frac{1}{x(1+\frac{1}{x})} \sim \sum_1^\infty \frac{(-1)^{n+1}}{x^n} \quad \text{as} \quad x \to \infty.$$

$$\frac{1}{x+1} = \frac{x-1}{x^2-1} \sim \sum_1^\infty \frac{(x-1)}{x^{2n}} \quad \text{as} \quad x \to \infty.$$

Also

$$\frac{1}{x+1} + e^{-x^2} \sim \sum_1^\infty \frac{(-1)^{n+1}}{x^n}.$$

ϕ, ψ are said to be asymptotically equivalent as $x \to x_0$ if

$$f(x) = g(x)(1 + O(1)).$$

Video clip covering above examples.

The usefulness of an asymptotic expansion arises from the fact that only a few terms of the series are required to give a good approximation to the function, whereas with a Taylor series expansion many terms are required for equivalent accuracy.

Note that from the definition of an asymptotic expansion, the remainder after N terms is much smaller than the last term retained as $x \to x_0$.

Example Consider

$$\mathrm{Ei}(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt.$$

Put $t = x + z$ and then

$$\mathrm{Ei}(x) = \frac{e^{-x}}{x} \int_0^\infty \frac{e^{-z}}{1+\frac{z}{x}}\, dz,$$

$$= \frac{e^{-x}}{x} \int_0^\infty e^{-z}\, dz \left[1 - \frac{z}{x} + \frac{z^2}{x^2} - + \cdots + \frac{(-1)^{n-1} z^{n-1}}{x^{n-1}} + \frac{(-1)^n z^n}{x^n(1+\frac{z}{x})}\right].$$

Integrating term by term gives

$$\mathrm{Ei}(x) = S_n(x) + R_n(x)$$

where

$$S_n(x) = e^{-x} \sum_{j=1}^{n} \frac{(-1)^{j+1}(j-1)!}{x^j},$$

$$R_n(x) = (-1)^n \frac{e^{-x}}{x} \int_0^\infty \frac{e^{-z} z^n}{x^n(1+\frac{z}{x})}\, dz = e^{-x} \int_0^\infty e^{-xt} \frac{(-1)^n t^n}{1+t}\, dt.$$

We have

$$|R_n(x)| < e^{-x} \int_0^\infty e^{-tx} t^n\, dt = e^{-x} \frac{n!}{x^{n+1}}.$$

Thus for fixed n, $R_n = O(\frac{e^{-x}}{x^{n+1}})$ as $x \to \infty$. Hence S_n is an asymptotic expansion for $\mathrm{Ei}(x)$ to n terms as $x \to \infty$.

Example Take $x = 10$

$$S_1(10) = 0.1 * e^{-10} \quad , |R_1(10)| < 0.01 * e^{-10}.$$

$$S_4(10) = 0.0914 * e^{-10}, \quad |R_4(10)| < 0.00024 * e^{-10}.$$

2.4 Additional notes

In general it is not permissible to differentiate asymptotic expansions.

Example If

$$f(x) = x + \sin x$$

then

$$f(x) \sim x \quad \text{as} \quad x \to \infty$$

but it is **not true** that

$$f'(x) \sim 1 \quad \text{as} \quad x \to \infty.$$

Example If

$$f(x) = e^{-x}\cos(e^x)$$

and x is real, then

$$f(x) \sim 0 + \frac{0}{x} + \frac{0}{x^2} + \dots \quad x \to \infty,$$

but

$$f'(x) = -\sin(x) - e^{-x}\cos(e^x)$$

oscillates as $x \to \infty$.

Differentiation is ok when it is known that $f'(x)$ is continuous and its asymptotic expansion exists. Also if $f(z)$ is an analytic function of z and has a Poincaré type of asymptotic power series expansion ie

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots \quad \text{to} \quad N \quad \text{terms as} \quad z \to \infty$$

uniformly in arg(z) in some sector S, then the expansion can be differentiated ie

$$f'(z) \sim -\frac{a_1}{z^2} + \frac{2a_2}{z^3} + \ldots \quad \text{to} \quad N-1 \quad \text{terms as} \quad z \to \infty$$

uniformly in arg(z) in some sector S' contained in S.

Integration is usually ok. Additional properties and proofs concerning asymptotic expansions may be found in [2], [3].

References

[1] Poincaré, H. 1886 Sur les intégrales irréguliéres des équations linéaires. Acta Math. 8, 259–344.

[2] Erdélyi, A. 1956. 'Asymptotic Expansions', Dover (reprint).

[3] Olver, F. W. J. 1924 'Asymptotics and Special Functions', AKP Classics (reprint).

3 Approximate solution of linear differential equations

3.1 Introduction

A large number of special functions are defined in terms of an ordinary differential equation. It is useful to be able to predict solution properties just by examining the coefficents of the differential operator. Fortunately, there exist powerful methods for predicting the local behaviour of the solutions near a point $x = x_0$ without needing to solve the full differential equation. In many cases the dominant behaviour can extracted without too much work, We will survey some of these ideas in this section.

3.2 Classification of singularities

Consider a homogeneous linear differential equation.

$$\mathcal{L}y = 0,$$

where

$$\mathcal{L} \equiv \frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \dots p_1(x)\frac{d}{dx} + p_0(x). \tag{3.1}$$

Definition - Ordinary Point *The point* $x = x_0(\neq \infty)$ *is called an* ordinary point *of (3.1) if* $p_0(x), p_1(x), \dots, p_{n-1}(x)$ *are analytic in a neighbourhood of* x_0*.*

Definition - regular singular point

The point $x = x_0(x_0 \neq \infty)$ *is a* regular singular point *of (3.1) if all of* $(x - x_0)^n p_0(x), (x - x_0)^{n-1} p_1(x), \dots, (x - x_0)p_{n-1}(x)$ *are analytic in a neighbourhood of* $x = x_0$*.*

Definition - irregular singular point *The point* $x = x_0(\neq \infty)$ *is called an* irregular singular point *of (3.1) if it is neither an ordinary point or a regular singular point. To classify the point at infinity, put* $x = 1/t$ *and rewrite the differential equation in terms of* t*. Then the point at* ∞ *is either an ordinary point, regular singular point, or an irregular singular point, if* $t = 0$ *is an ordinary point, regular singular point, or irregular singular point respectively.*

Examples

1. $y''(x) = (1 + x^2)y(x)$. Every point $x = x_0(\neq \infty)$ is an ordinary point.

2. $xy'''(x) - y'(x) + y = 0$. Every point $x = x_0$ with $x_0 \neq 0$ or $x_0 \neq \infty$ is an ordinary point.

3. $(x-1)y'''(x) + xy(x) = 0$ All points $x = x_0$, with $x_0 \neq 1$ or ∞ are ordinary points. $x_0 = 1$ is a regular singular point.

4. $x^3 y''(x) - y = 0$. The point $x = 0$ is not an ordinary point or a regular singular point.

Video clip of section on the O and o notation.

3.3 Properties near ordinary and regular singular points

All n linearly independent solutions of (3.1) are analytic in a neighbourhood of an ordinary point, [Fuchs (1866)]. The radius of convergence of a Taylor series of a solution about $x = x_0$ is at least as large as the distance to the nearest singularity of the coefficient functions. Near a regular singular point, the form of the n^{th} solution is at worst of the form,

$$y(x) = (x - x_0)^{\gamma} \sum_{k=0}^{n-1} [\log(x - x_0)]^k A_k(x)$$

where $A_k(x)$ is analytic at x_0, and γ is an *indicial exponent.*

Video clip of section on the O and o notation.

Example Consider Airy's equation

$$y'' = xy. \tag{3.2}$$

Here every point ($\neq \infty$) is an ordinary point and the solution can be expressed as a Taylor series expansion.

Seek a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and substitution into the equation (3.2) and equating coefficients of like powers of x leads to

$$a_n n(n-1) = 0, n = 0, 1, 2, \quad a_n n(n-1) = a_{n-3}, \quad n = 3, 4, \ldots$$

Thus a_1, a_2 are arbitrary, $a_2 = 0$ and

$$a_{3n} = \frac{a_0 \Gamma(\frac{2}{3})}{3^{2n} n! \Gamma(n + \frac{2}{3})}, \quad a_{3n+1} = \frac{a_1 \Gamma(\frac{4}{3})}{3^{2n} n! \Gamma(n + \frac{4}{3})}, \quad a_{3n+2} = 0.$$

The Gamma function $\Gamma(z)$ used above satisfies $\Gamma(z+1) = z\Gamma(z)$. Hence we have obtained two linearly independent solutions

$$y_1(x) = C_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n! \Gamma(n + \frac{2}{3})},$$

and

$$y_2(x) = C_1 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n! \Gamma(n + \frac{4}{3})}.$$

The radius of convergence of both series is infinity, the distance to the nearest singularity. By convention the two linearly independent solutions of Airy's equation, the Airy functions $\mathrm{Ai(x)}, \mathrm{Bi(x)}$ are defined by

$$\mathrm{Ai(x)} = 3^{-\frac{2}{3}} \sum_{\mathrm{n=0}}^{\infty} \frac{\mathrm{x}^{3\mathrm{n}}}{9^{\mathrm{n}} \mathrm{n}! \Gamma(\mathrm{n} + \frac{2}{3})} - 3^{-\frac{4}{3}} \sum_{\mathrm{n=0}}^{\infty} \frac{\mathrm{x}^{3\mathrm{n}+1}}{9^{\mathrm{n}} \mathrm{n}! \Gamma(\mathrm{n} + \frac{4}{3})},$$

$$\mathrm{Bi(x)} = 3^{-\frac{1}{6}} \sum_{\mathrm{n=0}}^{\infty} \frac{\mathrm{x}^{3\mathrm{n}}}{9^{\mathrm{n}} \mathrm{n}! \Gamma(\mathrm{n} + \frac{2}{3})} + 3^{-\frac{5}{6}} \sum_{\mathrm{n=0}}^{\infty} \frac{\mathrm{x}^{3\mathrm{n}+1}}{9^{\mathrm{n}} \mathrm{n}! \Gamma(\mathrm{n} + \frac{4}{3})}.$$

Video clip of section on the O and o notation.

3.4 Frobenius solution for 2nd order odes

Near a regular singular point the solution can be obtained as a Frobenius series in the form

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+\gamma}.$$

Here $a_0 \neq 0$ and γ is an indicial exponent (to be found), see below. Consider the equation

$$y''(x) + \bar{p}_1(x) y'(x) + \bar{p}_0(x) y(x) = 0, \tag{3.3}$$

and

$$\bar{p}_1(x) = \frac{1}{(x - x_0)} \sum_{n=0}^{\infty} p_n (x - x_0)^n, \quad \bar{p}_0(x) = \frac{1}{(x - x_0)^2} \sum_{n=0}^{\infty} q_n (x - x_0)^n.$$

If we seek a solution in Frobenius form

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+\gamma},$$

then substitution into the equation (3.3) gives:

$$[\gamma^2 + (p_0 - 1)\gamma + q_0] a_0 = 0, \tag{3.4}$$

$$[(\gamma + n)^2 + (p_0 - 1)(\gamma + n) + q_0] a_n = -\sum_{k=0}^{n-1} [(\gamma + k) p_{n-k} + q_{n-k}] a_k, \quad n = 1, 2, \ldots. \tag{3.5}$$

From (3.4) since $a_0 \neq 0$ we obtain the *indicial equation*

$$P(\gamma) \equiv \gamma^2 + (p_0 - 1)\gamma + q_0 = 0.$$

This gives two roots γ_1, γ_2, and we will assume that $\Re(\gamma_1) \leq \Re(\gamma_2)$. Then $P(\gamma_2 + n) \neq 0$ for $n = 1, 2, , \ldots$.

From (3.5) solving for a_n gives

$$a_n = -\frac{\sum_{k=0}^{n-1} [(\gamma + k) p_{n-k} + q_{n-k}] a_k}{P(\gamma + n)} \tag{3.6}$$

The expression (3.6) together with the fact that $P(\gamma_2 + n) \neq 0$ shows that we can obtain at least one solution in Frobenius form with the a_n given by (3.6) in terms of a_0 and $\gamma = \gamma_2$. Whether a second solution of this form exists or not, depends on whether the indicial roots differ by an integer or not. If $\gamma_2 - \gamma_1 \neq$ integer, then $P(\gamma + n) \neq 0$ and a second solution of Frobenius form also exists with a_n given by (3.6) in terms of a_0 and $\gamma = \gamma_1$. If $\gamma_2 - \gamma_1 = N$, where N is a positive integer then note that from (3.5) we obtain

$$P(\gamma_1 + N)a_N = -\sum_{k=0}^{N-1}[(\gamma_1 + k)p_{N-k} + q_{N-k}]a_k. \tag{3.7}$$

But $\gamma_1 + N = \gamma_2$ and thus the left hand side of(3.7) is

$$P(\gamma_2)a_N = 0.$$

If the right hand side of (3.7) equals zero then a_N is indeterminate and a second linearly independent solution of Frobenius type exists with $\gamma = \gamma_1$.

Video clip of section on the O and o notation.

Example Consider Rayleigh's equation which arises in Hydrodynamic Stability Theory, see MAGIC014.

$$\phi'' - (\alpha^2 + \frac{U''}{U-c})\phi = 0, \quad 0 < x < \infty,$$

where α and c are constants and $U = U(x)$. Suppose c is real and there exists x_c such that $U(x_c) = c$, and near $x = x_c$

$$U(x) = c + (x - x_c)U'(x_c) + \frac{1}{2}(x - x_c)^2 U''(x_c) + \dots.$$

Here $x = x_c$ is a regular singular point because in terms of our earlier notation in (3.3) $\bar{p}_1(x) = 0$ and

$$\bar{p}_0(x) = (\alpha^2 + \frac{U''}{U-c}) = \frac{q_1}{(x-x_c)} + q_2 + \dots, \quad \text{and} \quad q_1 = \frac{U''(x_c)}{U'(x_c)}.$$

The indicial equation is

$$\gamma(\gamma - 1) = 0, \quad \implies \gamma = 0, 1$$

and the roots differ by an integer.

Also the condition from (3.7) with $N = 1$ reduces to

$$q_1 = \frac{U''(x_c)}{U'(x_c)} = 0.$$

Thus if $U''(x_c) = 0$ then we have two linearly independent solutions of Frobenius type.

Video clip of section on the O and o notation.

3.5 Roots differ by an integer, $\gamma_2 - \gamma_1 = N$

Let

$$y(x, \gamma) = \sum_{n=0}^{\infty} a_n(\gamma)(x - x_0)^{\gamma+n}.$$

Now

$$\mathcal{L}y = a_0 P(\gamma)(x - x_0)^{\gamma-2} + \sum_{n=1}^{\infty} \left[a_n P(\gamma + n) + \sum_{j=0}^{n-1} (p_{n-j}(\gamma + j)a_j + q_{n-j}a_j) \right] (x - x_0)^{\gamma+n-2}. \tag{3.8}$$

Now let a_0 be arbitrary and choose $a_n(\gamma), \quad n = 1, 2, ...$ so that

$$a_n(\gamma) = -\frac{\sum_{k=0}^{n-1} [(\gamma + k)p_{n-k} + q_{n-k}]a_k}{P(\gamma + n)}$$

and assume that $P(\gamma + n) \neq 0$ for $n = 1, 2, ...$

3.5.1 Roots differ by an integer, $\gamma_2 - \gamma_1 = N$

Then from (3.8) we have

$$\mathcal{L}y = a_0 P(\gamma)(x - x_0)^{\gamma-2}. \tag{3.9}$$

We can see that if γ is chosen to be γ_2 the right hand side of (3.9) is zero and we have the solution $y(x, \gamma_2)$ obtained earlier.

3.5.2 Roots differ by an integer, $\gamma_2 - \gamma_1 = 0$

Suppose we differentiate both sides of (3.9) with respect to γ and then set $\gamma = \gamma_2$. Then

$$\begin{aligned}\mathcal{L}(\frac{\partial y}{\partial \gamma})|_{\gamma=\gamma_2} &= a_0((\gamma_2 - 2)\log(x-x_0)(x-x_0)^{\gamma_2-2}P(\gamma_2) \\ &+ (x-x_0)^{\gamma_2-2}P'(\gamma)).\end{aligned} \tag{3.10}$$

If the roots are equal ie $\gamma_2 - \gamma_1 = 0$ then $P'(\gamma_2) = 0$ and we see that the right hand side of (3.10) is zero. Therefore when we have equal roots a second linearly independent solution is

$$\frac{\partial y}{\partial \gamma}|_{\gamma=\gamma_2} = y(x,\gamma_2)\log(x-x_0) + \sum_{n=0}^{\infty} \frac{\partial a_n(\gamma)}{\partial \gamma}|_{\gamma=\gamma_2}(x-x_0)^{\gamma_2+n}.$$

Video clip of section on the O and o notation.

3.6 Roots differ by an integer $\gamma_2 - \gamma_1 = N > 0$

From (3.10) note that is we set $\gamma = \gamma_2$ the right hand side is equal to

$$a_0(x-x_0)^{\gamma_2-2}P'(\gamma_2) = a_0(x-x_0)^{\gamma_1+N-2}P'(\gamma_2),$$

and is not zero. However, consider

$$\mathcal{L}\left[(\frac{\partial y}{\partial \gamma})|_{\gamma=\gamma_2} + \sum_{n=0}^{\infty} b_n(x-x_0)^{\gamma_1+n}\right],$$

$$\begin{aligned}&= a_0(x-x_0)^{\gamma_1+N-2}P'(\gamma_2) + b_0P(\gamma_1)(x-x_0)^{\gamma_1-2} \\ &+ \sum_{n=1}^{\infty}[P(\gamma_1+n)b_n + \sum_{j=0}^{n-1}(p_{n-j}b_j + q_{n-j}b_j)](x-x_0)^{\gamma_1+n-2}.\end{aligned} \tag{3.11}$$

Equating powers of $(x - x_0)$ to zero gives:

$$\begin{aligned} P(\gamma_1)b_0 &= 0, & (3.12)\\ P(\gamma_1+n)b_n + \sum_{j=0}^{n-1}(p_{n-j}(\gamma_1+j)+q_{n-j})b_j &= 0, \quad n = 1, 2, .., N-1, \\ P(\gamma_1+n)b_n + \sum_{j=0}^{n-1}(p_{n-j}(\gamma_1+j)+q_{n-j})b_j &= 0, \quad n = N+1, \ldots & (3.13)\\ P(\gamma_1+N)b_N + \sum_{j=0}^{n-1}(p_{n-j}(\gamma_1+j)+q_{n-j})b_j &= a_0 P'(\gamma_2). & (3.14)\end{aligned}$$

From (3.12) since $P(\gamma_1) = 0$ we see that b_0 is undetermined.

But from (3.14) since $P(\gamma_1 + N) = P(\gamma_2)$ we have an expression which determines a_0 in terms of $b_0, b_1, \ldots, b_{N-1}$.

The term b_N is undetermined, but a non-zero b_N just replicates a multiple of the $y(x, \gamma_2)$ solution. Hence a second linearly independent solution is obtained in the form

$$y_1 = \frac{\partial y}{\partial \gamma}|_{\gamma=\gamma_2} + \sum_{n=0}^{\infty} b_n (x - x_0)^{\gamma_1+n}.$$

This can be expressed as

$$y_1 = k \log(x - x_0) y_2(x, \gamma_2) + \sum_{n=0}^{\infty} c_n (x - x_0)^{\gamma_1+n}. \qquad (3.15)$$

Note that if the right-hand side of (3.7) is zero, a_0 is zero and the coefficient k of the logarithmic term in (3.15) is zero.

Example Consider again Rayleigh's equation which we met in an earlier example:

$$\phi'' - (\alpha^2 + \frac{U''}{U - c})\phi = 0, \quad 0 < x < \infty,$$

where α and c are constants and $U = U(x)$. Suppose c is real and there exists x_c such that $U(x_c) = c$, and near $x = x_c$

$$U(x) = c + (x - x_c)U'(x_c) + \frac{1}{2}(x - x_c)^2 U''(x_c) + \ldots.$$

Here $x = x_c$ is a regular singular point because in terms of our earlier notation as in (3.3) $\bar{p}_1(x) = 0$ and

$$\bar{p}_0(x) = (\alpha^2 + \frac{U''}{U - c}) = \frac{q_1}{(x - x_c)} + q_2 + \ldots, \quad \text{and} \quad q_1 = \frac{U''(x_c)}{U'(x_c)}.$$

The indicial equation gives two roots $\alpha = 0$ and 1 differing by an integer. The Frobenius method gives two linearly independent solutions of the form

$$\phi_1(x) = (x - x_c) + a_2(x - x_c)^2 + a_3(x - x_c)^3 + \dots,$$

$$\phi_2(x) = 1 + b_1(x - x_c) + b_2(x - x_c)^2 + b_3(x - x_c)^3 + \dots$$
$$+ \frac{U''(x_c)}{U'(x_c)}\phi_1(x)(x - x_c)\log(x - x_c) \quad x > x_c.$$

The presence of the logarithmic branch point raises questions about what happens for $x < x_c$.

Video clip of worked example for Rayleigh's equation.

References

[Fuchs (1866)] Fuchs, L. (1866). Jour. für Math, **LXVI**, 121–160.

4 Approximate behaviour near an irregular singular point

We have seen how to construct the local solution properties near ordinary points and regular singular points. The more interesting case is to estimate behaviours near irregular singular points.

There is a powerful technique developed by [Carlini (1817)], [Liouville (1837)], [Green (1837)] based on the method of dominant balance. This is explained clearly with lots of illustrative examples in [Bender & Orszag (1999)]. Carlini's (1817) work concerned a problem in planetary motion. He introduced what is now known as the WKB expansion (see later in the course) and obtained an asymptotic expansion for a Bessel function of the first kind for large values of the parameter. Almost 20 years later [Liouville (1837)] used a similar WKB type expansion for a problem in heat conduction, and [Green (1837)] for a problem concerning waves in a fluid. The technique is more popularly known as the WKBJ after [Wentzel (1926)], [Kramers (1926)], [Brillouin (1926)], and [Jeffreys (1924)]. A historical account of the development of the WKBJ method can be found in [Pike (1964)], and [Fröman & Fröman (2002)].

Note that Frobenius type solutions do not work near irregular singular points. One example will suffice to illustrate this.

Example Consider
$$x^4 y'' = y,$$
and we see that $x = 0$ is an irregular singular point. If we look for a solution of the form
$$y = \sum_{n=0}^{\infty} a_n x^{n+\gamma}, \quad (a_0 \neq 0),$$
then we obtain
$$\sum_{n=0}^{\infty} a_n (n+\gamma)(n+\gamma-1) x^{n+\gamma+2} = \sum_{n=0}^{\infty} a_n x^{n+\gamma}.$$
The coefficient of x^γ gives $a_0 = 0$ which is a contradiction and therefore no solution of this type exists near $x = 0$.

4.1 Method of Dominant balance

The method of dominant balance relies on looking for local solutions of the form $y = e^{S(x)}$, as $x \to x_0$. The various steps are as follows.

- Substitute into the equation and retain only the dominant terms.
- Solve asymptotically for $S(x)$.

Video clip of worked example.

- Continue like this until the full leading order behaviour is obtained.
- Check that any assumptions made in the working are consistent.

We will illustrate the technique with an example.

Example Consider the equation

$$x^4 y'' = y \tag{4.1}$$

and look for a solution as $x \to 0$ of the form

$$y = e^{S(x)}.$$

Now

$$y'(x) = S'(x)e^{S(x)}, \quad y''(x) = (S'^2(x) + S''(x))e^{S(x)}. \tag{4.2}$$

Substitution of (4.2) into the equation (4.1) gives

$$x^4(S'^2 + S'') - 1 = 0. \tag{4.3}$$

We have to solve this for $S(x)$ as $x \to 0$. Let us assume that

$$S'(x) = cx^{\alpha} + \ldots, \quad S''(x) = c\alpha x^{\alpha-1} + \ldots.$$

Substitution into (4.3) gives

$$x^4(c^2x^{2\alpha} + c\alpha x^{\alpha-1}) \sim 1. \tag{4.4}$$

By balancing the various terms in (4.4) there appears to be various possibilities for choosing α. For example

- $c^2x^{4+2\alpha} \sim 1 \quad \Longrightarrow \alpha = -2.$
- $c^2x^{4+2\alpha} \sim -c\alpha x^{4+\alpha-1} \quad \Longrightarrow \alpha = -1.$
- $c\alpha x^{4+\alpha-1} \sim 1 \quad \Longrightarrow \alpha = -3.$

Only the first possibility is self consistent because choosing $\alpha = -1$ or 3 implies that the term omitted is larger than the ones retained for the balancing as $x \to 0$. Thus with $\alpha = -2$ and retaining the dominant terms gives

$$c^2 = 1, \quad \Longrightarrow c = \pm, 1.$$

We can continue in this manner and set

$$S'(x) = cx^{-2} + A_1(x), \tag{4.5}$$

where $A_1(x) = o(x^{-2})$. Substitution into (4.3) gives

$$x^4(c^2x^{-4} + 2cx^{-2}A_1 + A_1^2) + x^4(-2cx^{-3} + A_1') \sim 1,$$

or

$$2cx^2A_1 + x^4A_1^2 - 2cx + x^4A' \sim 0.$$

Again looking for a term of the form $A_1(x) = c_1x^\beta$ and looking for a dominant balance suggests that

$$\beta = -1, \quad c_1 = 1.$$

Other possibilities lead to inconsistencies.

Thus

$$S'(x) = cx^{-2} + x^{-1} + A_2(x), \quad A_2(x) = o(x^{-1}).$$

The equation for A_2 is

$$x^2(1 + 2cA_2) + 2x^3A_2 + x^4A_2^2 - x^2 + x^4A_2' \sim 0. \tag{4.6}$$

Note that (4.6) is identically satisfied by $A_2 = 0$ (not typical) giving

$$S'(x) = cx^{-2} + x^{-1}, \quad S(x) = -cx^{-1} + \log(x) + S_0.$$

Hence

$$y(x) = e^{S(x)} = Kxe^{\pm\frac{1}{x}}.$$

It can be verified that this satisfies the equation $x^4y'' = y$ exactly.

The previous example was unusual in that the expansion for $S(x)$ terminated after a finite number of terms. This is not typical.

Example Consider the equation

$$x^3y'' - y = 0. \tag{4.7}$$

Note that $x = 0$ is an irregular singular point of (4.7). We will seek a solution of the form $y = e^{S(x)}$ as $x \to 0$. This gives

$$x^3(S'^2 + S'') = 1. \tag{4.8}$$

Video clip of worked example for $y''-x^4y=0$'.

A dominant balance gives (with $c=\pm1$)

$$S'(x) = cx^{-\frac{3}{2}} + A(x), \quad A(x) = o(x^{-\frac{3}{2}}). \tag{4.9}$$

Substituting (4.9) into (4.8) gives

$$x^3(c^2x^{-3} + 2cx^{-\frac{3}{2}}A + A^2 - \frac{3}{2}cx^{-\frac{5}{2}} + A') \sim 1,$$

ie

$$2cx^{\frac{3}{2}}A + x^3A^2 - \frac{3c}{2}x^{\frac{1}{2}} + x^3A' \sim 0.$$

A dominant balance gives

$$A \sim \frac{3}{4x}.$$

Hence

$$S'(x) = cx^{-\frac{3}{2}} + \frac{3}{4}x^{-1} + B(x), \quad B(x) = o(x^{-1}). \tag{4.10}$$

The equation for B after substituting (4.10) into (4.8) is

$$\frac{9}{16}x + 2cBx^{\frac{3}{2}} + \frac{3}{2}x^2B + x^3B^2 - \frac{3}{4}x + x^3B' \sim 0.$$

This gives

$$B(x) = \frac{3}{32c}x^{-\frac{1}{2}} + o(x^{-\frac{1}{2}}).$$

Hence

$$S'(x) = cx^{-\frac{3}{2}} + \frac{3}{4}x^{-1} + \frac{3}{32c}x^{-\frac{1}{2}} + \dots,$$

giving

$$S(x) = -2cx^{-\frac{1}{2}} + \frac{3}{4}\log(x) - \frac{3}{16c}x^{\frac{1}{2}}.$$

Thus the leading order behaviour of $y(x)$ as $x \to 0$ is

$$y \sim e^{S(x)} \sim x^{\frac{3}{4}}e^{-2cx^{-\frac{1}{2}}}U(x),$$

Video clip showing leading order behaviour of $x^3y'' - y = 0$ as $x \to \infty$.

where $c = \pm 1$ and $U(x) = 1 + o(x^{\frac{1}{2}})$.

The above gives the leading order asymptotic behaviour of the the solutions. The full asymptotic behaviour for $y(x)$ requires more work. To do this we first set

$$y(x) = e^{2cx^{-\frac{1}{2}}} W(x), \quad W(x) \sim \sum_{n=0}^{\infty} a_n x^{n\alpha + \frac{3}{4}}, \tag{4.11}$$

where α is to be found. We have

$$y' = [-cx^{-\frac{3}{2}} W + W'] e^{2cx^{-\frac{1}{2}}},$$

$$y'' = [c^2 x^{-3} W - 2cx^{-\frac{3}{2}} W' + \frac{3c}{2} x^{-\frac{5}{2}} W + W''] e^{2cx^{-\frac{1}{2}}}.$$

Substituting into the equation $x^3y'' - y = 0$ gives

$$W'' - 2cx^{-\frac{3}{2}} W' + \frac{3c}{2} x^{-\frac{5}{2}} W = 0. \tag{4.12}$$

If we seek an asymptotic expansion for $W(x)$ as $x \to 0$ in the form

$$W(x) \sim \sum_{n=0}^{\infty} a_n x^{n\alpha + \frac{3}{4}},$$

with ($a_0 \neq 0$) then substitution into (4.12) gives

$$\sum_{n=0}^{\infty} a_n (n\alpha + \frac{3}{4})(n\alpha + \frac{3}{4} - 1) x^{n\alpha + \frac{3}{4} - 2} - 2c \sum_{n=0}^{\infty} a_n (n\alpha + \frac{3}{4}) x^{n\alpha + \frac{3}{4} - \frac{5}{2}} + \frac{3c}{2} \sum_{n=0}^{\infty} a_n x^{n\alpha + \frac{3}{4} - \frac{5}{2}} \sim 0. \tag{4.13}$$

Note that the coefficient of the dominant term $x^{\frac{3}{4} - \frac{5}{2}}$ in (4.13) is

$$a_0 (-\frac{3c}{2} + \frac{3c}{2}) = 0,$$

which is satisfied identically leaving a_0 undetermined. Balancing the next terms in (4.13) suggests that

$$x^{-2} \sim x^{\alpha - \frac{5}{2}}$$

giving $\alpha = \frac{1}{2}$.

The coefficient of $x^{\frac{3}{4}+n\alpha-2}$ in (4.13) shows that

$$a_n(n\alpha + \frac{3}{4})(n\alpha + \frac{3}{4} - 1) - 2ca_{n+1}(n+1)\alpha = 0, \quad n = 0, 1, 2, \ldots$$

giving

$$a_{n+1} = \frac{(2n+3)(2n-1)}{16c(n+1)} a_n, \quad n = 0, 1, 2, \ldots.$$

Hence an asymptotic expansion of the solution to

$$x^3 y'' - y = 0$$

as $x \to 0$ is

$$y \sim Ae^{2cx^{-\frac{1}{2}}} x^{\frac{3}{4}} (1 - \frac{3}{16c} x^{\frac{1}{2}} + \ldots a_n x^{\frac{n}{2}} + \ldots),$$

where A is an arbitrary constant, $c = \pm 1$ and $a_0 = 1$

$$a_{n+1} = \frac{(2n+3)(2n-1)}{16c(n+1)} a_n, \quad n = 0, 1, 2, \ldots.$$

Video clip of example showing full asymptotic expansion for $x^3 y'' = y$. as $x \to \infty$.

Example Consider the equation

$$x^2 y'' + xy' - (x^2 + \nu^2) y = 0. \tag{4.14}$$

Note that $x = \infty$ is an irregular singular point. The leading order behaviour is easily obtained to be

$$y(x) \sim Cx^{-\frac{1}{2}} e^{\pm x} \quad \text{as} \quad x \to \infty.$$

We will obtain the full asymptotic behaviour as $x \to \infty$. Write

$$y(x) = Ae^{cx} W(x),$$

where

$$W(x) = x^{-\frac{1}{2}}(1 + o(x)), \quad \text{as} \quad x \to \infty,$$

A is an arbitrary constant and $c = \pm 1$. If we substitute into the equation (4.14) we find that W satisfies

$$x^2 W'' + (2cx^2 + x)W' + (cx - \nu^2)W = 0.$$

We seek an asymptotic expansion of $W(x)$ as

$$W(x) \sim \sum_{n=0}^{\infty} a_n x^{n\alpha - \frac{1}{2}},$$

with $a_0 \neq 0$. Substitution into the equation for W gives

$$\sum_{n=0}^{\infty} a_n(n\alpha - \frac{1}{2})(n\alpha - \frac{3}{2})x^{n\alpha-\frac{1}{2}} + \sum_{n=0}^{\infty} a_n(n\alpha - \frac{1}{2})(2cx^{n\alpha+\frac{1}{2}} + x^{n\alpha-\frac{1}{2}}) + \sum_{n=0}^{\infty} a_n(cx^{n\alpha+\frac{1}{2}} - \nu^2 x^{n\alpha-\frac{1}{2}}) \sim 0. \tag{4.15}$$

The coefficient of the a_0 term in (4.15) is zero. The dominant balance in (4.15) suggests that

$$x^{-\frac{1}{2}} \sim x^{\alpha+\frac{1}{2}} \qquad \Longrightarrow \alpha = -1.$$

Equating the coefficients of $x^{n\alpha-\frac{1}{2}}$ in (4.15) to zero gives

$$(n\alpha - \frac{1}{2})(n\alpha - \frac{3}{2})a_n + a_{n+1}((n+1)\alpha - \frac{1}{2})2c + a_n(n\alpha - \frac{1}{2}) + a_{n+1}c - \nu^2 a_n = 0.$$

Hence

$$a_n[(n + \frac{1}{2})^2 - \nu^2] - 2c(n+1)a_{n+1} = 0, \quad n = 0, 1, \ldots\ldots,$$

giving

$$a_{n+1} = \frac{(n+\frac{1}{2})^2 - \nu^2}{2c(n+1)}, \quad n = 0, 1, \ldots. \tag{4.16}$$

The solutions of (4.14) therefore have the behaviour

$$y(x) \sim Ax^{-\frac{1}{2}}e^{cx}(1 + \sum_{n=1}^{\infty} a_n \frac{1}{x^n})$$

as $x \to \infty$, with $a_0 = 1$ and a_n given by (4.16).

Note that the series terminates if

$$\nu = \pm(n + \frac{1}{2}) \quad n = 0, 1, \ldots,$$

in which case we have an exact solution of the equation.

4.2 Airy Functions

Airy functions arise often in asymptotic expansions and in the theory of differential equations. We will look at a few properties.

$\mathrm{Ai}(z), \mathrm{Bi}(z)$ are called Airy functions and are two linearly independent solutions of

$$y'' - zy = 0. \tag{4.17}$$

Note that every point $\neq \infty$ is an ordinary point of the differential equation, and if we look for a Taylor series solution we find

$$y = \sum_{n=0}^{\infty} a_n z^n, \quad a_0 \neq 0$$

and

$$a_{3n} = \frac{\Gamma(\frac{2}{3})}{9^n n! \Gamma(n+\frac{2}{3})} a_0, a_{3n+1} = \frac{\Gamma(\frac{4}{3})}{9^n n! \Gamma(n+\frac{4}{3})} a_1, a_{3n+2} = 0, \tag{4.18}$$

where a_0, a_1 are arbitrary constants - see lecture 2. Thus

$$y(z) = a_0 \Gamma(\frac{2}{3}) \sum_{n=0}^{\infty} \frac{z^{3n}}{9^n n! \Gamma(n+\frac{2}{3})} + a_1 \Gamma(\frac{4}{3}) \sum_{n=0}^{\infty} \frac{z^{3n+1}}{9^n n! \Gamma(n+\frac{2}{3})}.$$

The radius of convergence of the series is infinite since all points are ordinary points. We define $\mathrm{Ai}(z), \mathrm{Bi}(z)$ by

$$\mathrm{Ai(z)} = 3^{-2/3} \sum_{n=0}^{\infty} \frac{z^{3n}}{9^n n! \Gamma(n+\frac{2}{3})} - 3^{-4/3} \sum_{n=0}^{\infty} \frac{z^{3n+1}}{9^n n! \Gamma(n+\frac{4}{3})} \tag{4.19}$$

$$\mathrm{Bi}(z) = 3^{-1/6} \sum_{n=0}^{\infty} \frac{z^{3n}}{9^n n! \Gamma(n+\frac{2}{3})} + 3^{-5/6} \sum_{n=0}^{\infty} \frac{z^{3n+1}}{9^n n! \Gamma(n+\frac{4}{3})}. \tag{4.20}$$

For large $x \to \infty$

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}},$$

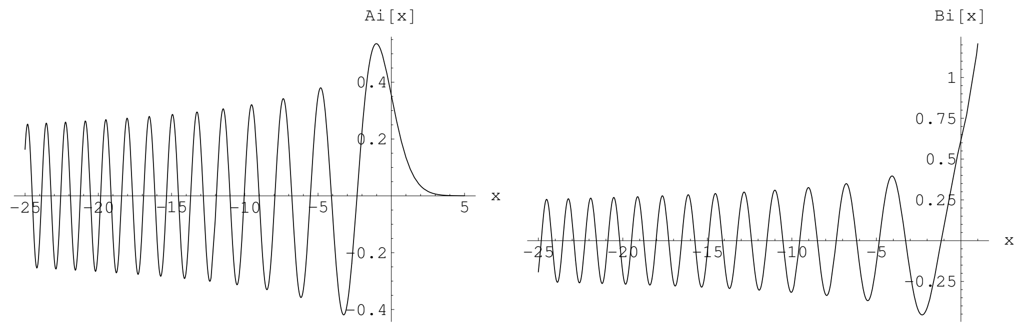
$$\mathrm{Bi}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{\frac{2}{3}x^{\frac{3}{2}}}.$$

A study of the large x behaviour of the differential equation yields

$$y(x) \sim C x^{-\frac{1}{4}} e^{\pm x^{\frac{3}{2}}},$$

but the constants appropriate to $\mathrm{Ai}(x), \mathrm{Bi}(x)$ can only be determined from an integral representation of the functions. For $x \to -\infty$ we can look for a solution of the form $y = e^{S(x)}$ as before. This leads to

$$S'' + S'^2 - x = 0. \tag{4.21}$$



(a) (b)

Figure 1: Sample plots of the Airy function (a) $\mathrm{Ai}(x)$ and (b) $\mathrm{Bi}(x)$ on the real line. Notice the highly oscillatory behaviour for large negative x. $\mathrm{Ai}(x)$ decays exponentially for large positive x and $\mathrm{Bi}(x)$ grows exponentially for large positive x.

Hence

$$S'(x) \sim \pm i(-x)^{\frac{1}{2}}, \quad S(x) \sim \pm\frac{2}{3}i(-x)^{\frac{3}{2}} \quad \text{as} \quad x \to -\infty.$$

Writing

$$S = \pm i\frac{2}{3}(-x)^{\frac{3}{2}} + B(x), \quad B(x) = o((-x)^{\frac{3}{2}}),$$

we find that after substitution into (4.21) that

$$B(x) \sim -\frac{1}{4}\log(-x).$$

Hence

$$y(x) \sim C(-x)^{-\frac{1}{4}}e^{\pm\frac{2}{3}i(-x)^{\frac{3}{2}}} \quad \text{as} \quad x \to -\infty.$$

4.3 Airy functions, behaviour for large x.

Since $\mathrm{Ai}(x), \mathrm{Bi}(x)$ are real for real arguments x the behaviour as $x \to -\infty$ must be a linear combination of the above solutions. Hence

$$y(x) \sim C_1(-x)^{-\frac{1}{4}}\sin(\frac{2}{3}(-x)^{\frac{3}{2}}) + C_2(-x)^{-\frac{1}{4}}\cos(\frac{2}{3}(-x)^{\frac{3}{2}}), \tag{4.22}$$

as $x \to -\infty$.

Further terms in the expansion may be obtained by writing

$$y(x) \sim C_1(-x)^{-\frac{1}{4}}w_1(x)\sin(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}) + C_2(-x)^{-\frac{1}{4}}w_2(x)\cos(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}), \tag{4.23}$$

where the $\pi/4$ factor is inserted for convenience. After substitution into the equation (4.17) one can find the behaviours of $w_1(x), w_2(x)$. It is convenient to introduce $t = -x$ and rewrite (4.23) as

$$y(x) \sim W_1(t)\sin(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}) + W_2(t)\cos(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}), \qquad (4.24)$$

Then

$$\frac{dy}{dt} = [W_1 t^{\frac{1}{2}} + W_2']\cos(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}) + [W_1' - W_2 t^{\frac{1}{2}}]\sin(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}).$$

$$\frac{d^2y}{dt^2} = [-t^{\frac{1}{2}}(W_1 t^{\frac{1}{2}} + W_2') + W_1'' - \frac{1}{2}t^{-\frac{1}{2}}W_2 - W_2' t^{\frac{1}{2}}]\sin(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4})$$
$$+[t^{\frac{1}{2}}(W_1' - W_2 t^{\frac{1}{2}}) + W_2'' + \frac{1}{2}t^{-\frac{1}{2}}W_1 + W_1' t^{\frac{1}{2}}]\cos(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4}).$$

Hence substituting into Airy's equation in terms of t ie

$$\frac{d^2y}{dt^2} + ty = 0,$$

and equating the coefficents of the sine and cosine terms to zero leads to

$$W_1'' - 2t^{\frac{1}{2}}W_2' - \frac{1}{2}t^{-\frac{1}{2}}W_2 = 0, \qquad (4.25)$$

$$W_2'' + 2t^{\frac{1}{2}}W_1' + \frac{1}{2}t^{-\frac{1}{2}}W_1 = 0. \qquad (4.26)$$

Next seek asymptotic expansion solutions to these equations in the form

$$W_1(t) = \sum_{n=0}^{\infty} a_n t^{-n\alpha-\frac{1}{4}}, \quad W_2(t) = \sum_{n=0}^{\infty} b_n t^{-n\beta-\frac{1}{4}}.$$

The equation (4.25) for W_1 leads to

$$\sum_{n=0}^{\infty} a_n(n\alpha + \frac{1}{4})(n\alpha + \frac{1}{4} + 1)t^{-n\alpha} + \sum_{n=0}^{\infty} b_n(2(n\beta + \frac{1}{4}) - \frac{1}{2})t^{-n\beta+\frac{3}{2}} = 0, \quad (4.27)$$

and the equation (4.26)for W_2 to

$$\sum_{n=0}^{\infty} b_n(-n\beta + \frac{1}{4})(n\beta + \frac{1}{4} + 1)t^{-n\beta} + \sum_{n=0}^{\infty} a_n(-2(n\alpha + \frac{1}{4}) + \frac{1}{2})t^{-n\alpha+\frac{3}{2}} = 0, \quad (4.28)$$

The dominant terms in (4.27,4.28) show that

$$0.b_0 = 0, \quad 0.a_0 = 0$$

leaving a_0, b_0 arbitrary.

At next order we obtain $\beta = \alpha = \frac{3}{2}$ and

$$b_{n+1} = -\frac{(6n+1)(6n+5)}{48(n+1)} a_n, \quad a_{n+1} = \frac{(6n+1)(6n+5)}{48(n+1)} b_n.$$

The choice of the constants

$$a_0 = \frac{1}{\sqrt{\pi}}, \quad b_0 = 0$$

represents the behaviour of $\mathrm{Ai}(x)$ as $x \to -\infty$. In this case the terms

$$a_{2n+1} = 0, n = 0, 1, \ldots, \quad b_{2n} = 0, n = 0, 1, \ldots.$$

We obtain

$$\mathrm{Ai}(x) \sim (-x)^{-\frac{1}{4}} w_1(x) \sin[\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}] + (-x)^{-\frac{1}{4}} w_2(x) \cos[\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}]$$

where

$$w_1(x) \sim \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} c_{2n} x^{-3n}, \quad x \to -\infty,$$

$$w_2(x) \sim -\frac{1}{\sqrt{\pi}} (-x)^{-\frac{3}{2}} \sum_{n=0}^{\infty} c_{2n+1} x^{-3n} \quad x \to -\infty,$$

and

$$c_n = \frac{1}{2\pi} \left(\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}.$$

The behaviour of $\mathrm{Bi}(x)$ is described by the choice

$$a_0 = 0, b_0 = \frac{1}{\sqrt{\pi}}.$$

In this case the terms

$$a_{2n} = 0, \quad n = 0, 1, \ldots, \quad b_{2n+1} = 0, \quad n = 0, 1, \ldots.$$

We obtain

$$\mathrm{Bi}(x) \sim (-x)^{-\frac{1}{4}} w_1(x) \sin[\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}] + (-x)^{-\frac{1}{4}} w_2(x) \cos[\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}]$$

where

$$w_1(x) \sim \frac{1}{\sqrt{\pi}} (-x)^{-\frac{3}{2}} \sum_{n=0}^{\infty} c_{2n+1} x^{-3n}, \quad x \to -\infty,$$

$$w_2(x) \sim \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} c_{2n} x^{-3n} \quad x \to -\infty,$$

and c_n are as before.

Video clip for discussion of the properties of Airy's functions.

4.4 Stokes's Phenomenon

If we write

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_0$$

then it is unclear which path we are specifying as $z \to z_0$ in the complex plane.

For the equation

$$\frac{d^2y}{dz^2} - zy = 0$$

we found that

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}}, \quad \mathrm{Bi}(z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} e^{+\frac{2}{3}z^{\frac{3}{2}}}. \tag{4.29}$$

But Ai(z) is an enitre function and its Taylor series (reftayairyai) converges for all finite values of $|z|$ whereas the right-hand side of (4.29) is a multi-valued function with branch points. How is this resolved. Note that

$$f(z) \sim g(z)$$

holds only in a certain sector. Since Bi(z) grows exponentially along the real axis it suggests to restrict z such that

$$|\arg(z^{\frac{3}{2}})| < \frac{\pi}{2}, \quad \Longrightarrow |\arg(z)| < \frac{\pi}{3}.$$

Thus the sector of validity for Bi(z) to have the behaviour as in (4.29) is $|\arg(z)| < \pi/3$.

In general if

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_0$$

then

$$f(z) - g(z) = o(g(z)) \quad \text{as} \quad z \to z_0.$$

Now

$$f(z) = g(z) + (f(z) - g(z)).$$

We say that when z lies in a certain sector $g(z)$ is *dominant* and $f(z)-g(z)$ small or *subdominant*. As the edges of the sector, or wedge, are approached $f(z)-g(z)$ is not small. Outside the sector $f-g$ becomes larger than g. This exchange of identities is called *Stokes's Phenomenon* after Stokes(1857) who first observed it.

The edges of the sector or wedge where the difference in behaviour occurs on different sides are called *Stokes Lines.* For some of the second order equations studied earlier, we observed that

$$y \sim e^{S_{1,2}(x)} \quad \text{as} \quad z \to z_0.$$

Stokes lines are defined by

$$\Re(S_1(z) - S_2(z)) = 0.$$

and *anti-Stokes* lines by

$$\Im(S_1(z) - S_2(z)) = 0.$$

Example Consider the Airy functions. The Stokes lines are given by

$$Re(z^{\frac{3}{2}}) = 0$$

giving

$$\arg(z) = \pm\frac{\pi}{3}, \pi, \quad |z| \to \infty.$$

The function $\mathrm{Bi}(z)$ has the behaviour

$$\mathrm{Bi}(z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3}z^{\frac{3}{2}}}$$

valid only in the sector $|\arg(z)| < \pi/3$.

However for $\mathrm{Ai}(z)$ it can be shown from the integral representation (see below) for $\mathrm{Ai}(z)$ that

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}},$$

as $z \to \infty$ holds in a much larger sector for $|\arg(z)| < \pi$.

Video clip for discussion of the Stokes' phenomenon.

4.5 Linear relations between Airy functions

In the equation

$$\frac{d^2y}{dz^2} - zy = 0$$

we can replace zy by $\omega^3 zy$ where $\omega = e^{-2i\pi/3}$ is a cube root of unity.

Next put $t = \omega z$ and note that

$$\frac{d^2y}{dt^2} - ty = 0$$

so that $y = \text{Ai}(\omega z)$ is also a solution of Airy's equation. Similarly $\text{Ai}(z), \text{Ai}(\omega z)$,$\text{Ai}(\omega^2 z), \text{Bi}(z)$ are all solutions of Airy's equation but we can only have two linearly independent solutions. Hence there exists a, b such that

$$\text{Ai}(z) = a\text{Ai}(\omega z) + b\text{Ai}(\omega^2 z).$$

From the Taylor series (4.19) for $\text{Ai}(z)$ comparing the coefficents of the z^0, z terms shows that

$$a + b = 1, \quad a\omega + b\omega^2 = 1.$$

Hence

$$a = -\omega, \quad b = -\omega^2.$$

Thus

$$\text{Ai}(z) = -\omega\text{Ai}(\omega z) - \omega^2\text{Ai}(\omega^2 z), \tag{4.30}$$

and similarly

$$\text{Bi}(z) = i\omega\text{Ai}(\omega z) - i\omega^2\text{Ai}(\omega^2 z). \tag{4.31}$$

These relations can be used to obtain asymptotic expansions for $\text{Ai}(z), \text{Bi}(z)$ valid in other sectors given that

$$\text{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}} \sum_{n=0}^{\infty} (-1)^n c_n z^{-\frac{3n}{2}}, \quad |\arg(z)| < \pi. \tag{4.32}$$

with

$$c_n = \frac{1}{2\pi}\left(\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}.$$

To use (4.32) with (4.30) we require that

$$-\pi < \arg(\omega z) < \pi, \quad \text{and} \quad -\pi < \arg(\omega^2 z) < \pi.$$

This implies that provided $\pi/3 < \arg(z) < 5\pi/3$, we can write

$$\text{Ai}(z) \sim -\omega \left[\frac{1}{2\sqrt{\pi}} (\omega z)^{-\frac{1}{4}} e^{-\frac{2}{3}(\omega z)^{\frac{3}{2}}} \sum_{n=0}^{\infty} (-1)^n c_n (\omega z)^{-\frac{3n}{2}} \right]$$

$$-\omega^2\left[\frac{1}{2\sqrt{\pi}}(\omega^2 z)^{-\frac{1}{4}}e^{-\frac{2}{3}(\omega^2 z)^{\frac{3}{2}}}\sum_{n=0}^{\infty}(-1)^n c_n(\omega^2 z)^{-\frac{3n}{2}}\right].$$

Hence for $\pi/3 < \arg(z) < 5\pi/3$

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}}z^{-\frac{1}{4}}e^{-\frac{2}{3}z^{\frac{3}{2}}}\sum_{n=0}^{\infty}(-1)^n c_n z^{-\frac{3n}{2}} + \frac{i}{2\sqrt{\pi}}z^{-\frac{1}{4}}e^{\frac{2}{3}z^{\frac{3}{2}}}\sum_{n=0}^{\infty}(-1)^n c_n z^{-\frac{3n}{2}}.$$

Video clip for properties of Airy functions and asymptotic expansions across different sectors.

4.6 Integral represensations of Airy functions

Consider the Airy equation

$$\frac{d^2y}{dz^2} - zy = 0$$

and suppose we seek a solution in the form

$$y(z) = \int_c F(s)e^{sz}\,ds.$$

Substitution into the equation shows that

$$\int_c (s^2 - z)F(s)e^{sz}\,ds = 0.$$

Integrate by parts then

$$[-F(s)e^{sz}]_C + \int_C (s^2 F + \frac{dF}{ds})e^{sz}\,ds = 0.$$

The first term above is to be evaluated at the endpoints of the curve C. Suppose we choose F so that

$$\frac{dF}{ds} + s^2 F = 0,$$

ie

$$F(s) = e^{-\frac{s^3}{3}}.$$

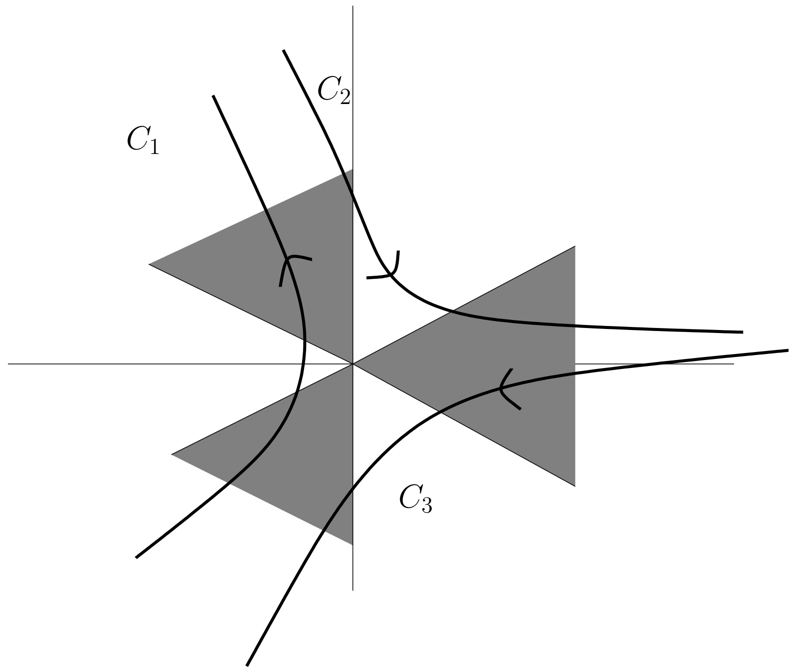


Figure 2: Various contours for solutions of the Airy equation in the integral representation

For this to satisfy the equation we also need to choose a suitable contour C so that

$$[F(s)e^{sz}]_C = [e^{-\frac{s^3}{3}+sZ}]_C = 0.$$

This gives rise to three sectors

$$-\frac{\pi}{6} < \arg(s) < \frac{\pi}{6}, \quad \frac{\pi}{2} < \arg(s) < \frac{5\pi}{6}, \quad -\frac{\pi}{2} < \arg(s) < -\frac{5\pi}{6}$$

where $|e^{-\frac{s^3}{3}}| \to 0$, provided the endpoints of the start and begin in these sectors. This gives rise to three functions

$$f_n = \frac{1}{2\pi i}\int_{C_n} e^{sz-\frac{s^3}{3}}\,ds,$$

where the curves are as in the fig. 2. The Airy function $Ai(z)$ is given by

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_{C_1} e^{sz-\frac{s^3}{3}}\,ds.$$

The Airy function of the second kind $\mathrm{Bi}(z)$ is given by

$$\mathrm{Bi}(z) = i[f_2(z) - f_3(z)].$$

Video clip for discussion of the integral representation of Airy's functions.

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5 Matched expansions, Boundary Layer Theory, WKB method.

5.1 Regular and singular perturbation problems.

In this section we will consider boundary layer and WKB theory for obtaining asymptotic solutions to differential equations whose highest derivatives are multiplied by a small parameter ϵ. We will find that the solutions change rapidly in thin regions as $\epsilon \to 0$. A **singular perturbation** problem is characterised by the fact that the $\epsilon = 0$ problem has quite different solution properties as compared to the $0 < \epsilon << 1$ problem. In a **regular perturbation** problem as $\epsilon \to 0$ the solution tends to the solution for $\epsilon = 0$. This is best illustrated by looking at some simple examples.

Example

Consider

$$y'' + 2\epsilon y' - y = 0, \quad y(0) = 0, \quad y(1) = 1 \tag{5.1}$$

and $0 < \epsilon << 1$. The general solution is

$$y(x, \epsilon) = \frac{e^{m_1 x} - e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

where

$$m_1 = -\epsilon + \sqrt{1 + \epsilon^2}, \quad m_2 = -\epsilon - \sqrt{1 + \epsilon^2}.$$

As $\epsilon \to 0$ we have

$$y(x) \to \frac{\sinh(x)}{\sinh(1)},$$

and everything seems ok.

We can also obtain a solution as follows: Write

$$y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \dots.$$

Substitution into the equation (5.1) and equating coefficents of like powers of ϵ to zero gives

$$\begin{aligned} Y_0'' - Y_0 &= 0, \quad Y_0(0) = 0, \quad Y_0(1) = 1 \\ Y_1'' - Y_1 &= -2Y_0', \quad Y_1(0) = 0, Y_1(1) = 0. \end{aligned} \tag{5.2}$$

Solving (5.2) gives

$$Y_0 = \frac{\sinh(x)}{\sinh(1)}, \quad Y_1 = (1 - x)\frac{\sinh(x)}{\sinh(1)}.$$

Again there are no problems - we have a regular perturbation problem.

Video clip for regular perturbation problem example.

** check if publically viewable*

Example

Consider

$$\epsilon y'' + 2y' - y = 0, \quad y(0) = 0, \quad y(1) = 1, \tag{5.3}$$

for $0 < \epsilon << 1$. The solution is as before

$$y(x, \epsilon) = \frac{e^{m_1 x} - e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

where now

$$m_1 = \frac{1}{\epsilon}(-1 + \sqrt{1+\epsilon}), \quad m_2 = \frac{1}{\epsilon}(-1 - \sqrt{1+\epsilon}).$$

As $\epsilon \to 0$ we have

$$m_1 \to \frac{1}{2}, \quad m_2 \sim -\frac{2}{\epsilon}.$$

Note that as $\epsilon \to 0$

$$y \sim \frac{1}{(e^{\frac{1}{2}} - e^{-\frac{2}{\epsilon}})}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}) \sim e^{-\frac{1}{2}}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}).$$

Clearly there are two distinct regions:

- $\frac{x}{\epsilon} = O(1)$, and then
$$y \sim e^{-\frac{1}{2}}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}).$$
- $x >> \epsilon$ and then
$$y \sim e^{-\frac{1}{2}} e^{\frac{x}{2}}.$$

The analytic solution for different values of ϵ is shown in Fig. 3. Note that the solution changes rapidly in the region $x = O(\epsilon)$. We have an example of a singular limit as $\epsilon \to 0$. The region $x = O(\epsilon)$ is called a *boundary layer.*

Suppose we try solving the equation as before. Put

$$y = Y_0 + \epsilon Y_1 + \ldots.$$

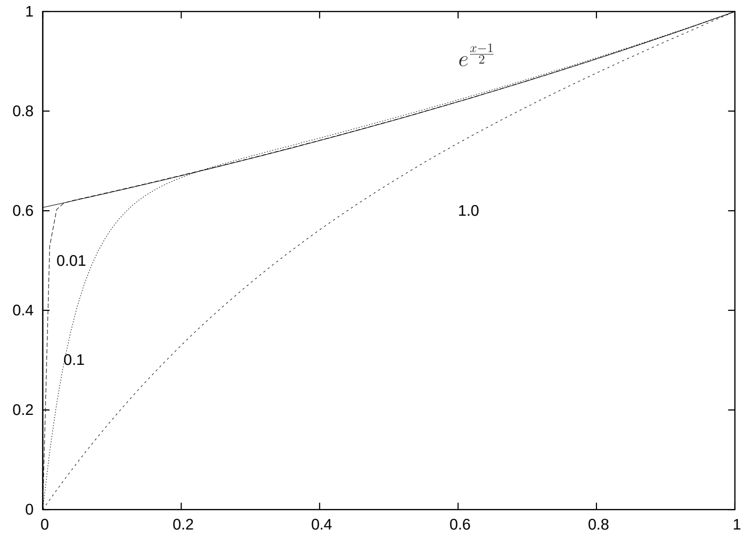

Figure 3: Solution $y(x, \epsilon)$ for different values of ϵ.

This gives after substitution into (5.3)

$$2Y_0' - Y_0 = 0, \quad 2Y_1' - Y_1 = Y_0'', \tag{5.4}$$

and boundary conditions

$$Y_0(0) = 0, \quad Y_0(1) = 1,$$

$$Y_1(0) = 0, \quad Y_1(1) = 0,$$

etc. Now there is a problem! The order of the equations (5.4) is reduced, ie we now have first order equations for the Y_i. Consequently which boundary conditions do we choose? The exact solution suggests we can satisfy the condition at $x = 1$. Let us continue with the boundary condition at $x = 1$.

Video clip showing working for singular perturbation example.

Solution of first order problem

$$2Y_0' - Y_0 = 0, \quad Y_0(1) = 1,$$

gives

$$Y_0 = e^{\frac{x-1}{2}}.$$

Clearly this solution is not valid for all x since the condition at $x = 0$ is not satisfied. When x is small the solution fails and we need to examine this region in more detail. The Y_0 solution is the leading order *outer solution*. Now when x is small we have

$$Y_0 \sim e^{-\frac{1}{2}}(1 + \frac{x}{2}) = O(1).$$

Put $x = \epsilon^n X$ say where $n > 0$ is to be found. The variable X is called the inner variable and is $O(1)$ in the *inner region* of thickness $O(\epsilon^n)$. The differential equation (5.3) in terms of X is

$$\epsilon^{1-2n} \frac{d^2 y}{dX^2} + 2\epsilon^{-n} \frac{dy}{dX} - y = 0. \tag{5.5}$$

For $n > 0$ the dominant terms are the first two terms and these balance if

$$1 - 2n = -n \quad \Longrightarrow n = 1.$$

A quick consistency check shows that this is ok, (other choices for n eg $n = 1/2$ are not). In the inner region if we put

$$y = y_0(X) + \epsilon^\alpha y_1(X) + \dots$$

with $\alpha > 0$ and substitute into (5.5) (with $n = 1$) we find that the leading order problem is

$$\frac{d^2 y_0}{dX^2} + 2\frac{dy_0}{dX} = 0,$$

and one boundary condition is $y_0(X = 0) = 0$.

The other condition must come from *matching* with the outer solution taking X large. Solving yields

$$y_0(X) = A + Be^{-2X}$$

and $y_0(0) = 0$ implies that $A = -B$. Thus

$$y_0(X) = A(1 - e^{-X}).$$

To obtain the constant A we match the inner solution just derived with the outer solution.

$$y_0(X) \sim A \quad \text{for} \quad X >> 1,$$

and

$$Y_0(x) \sim e^{-\frac{1}{2}} \quad \text{for} \quad x \to 0.$$

This gives $A = e^{-\frac{1}{2}}$ and

$$y_0(X) = e^{-\frac{1}{2}}(1 - e^{-2X}).$$

Summary so far: 1 term inner and 1 term outer expansions.

outer:

$$x = O(1), \quad y = Y_0(x) + \epsilon Y_1(x) + \dots,$$
$$Y_0(x) = e^{-\frac{1}{2}} e^{\frac{x}{2}}.$$

inner:

$$x = \epsilon X, \quad y = y_0(X) + \epsilon^{\alpha} y_1(X) + \dots,$$
$$y_0(X) = e^{-\frac{1}{2}}(1 - e^{-2X}).$$

Video clip for discussion boundary layer solution.

These are the basics of boundary layer theory and matched asymptotic expansions. The solution can be continued to higher order. Notice that the outer solution expanded for small x gives

$$y \sim e^{-\frac{1}{2}}(1 + \frac{x}{2} + \dots) + \epsilon Y_1(x) + \dots.$$

When written in terms of $x = \epsilon X$ this suggests that the inner solution should proceed as

$$y = y_0 + \epsilon y_1 + \dots.$$

We had assumed that the outer expansion proceeded in powers of ϵ but this does not have to be the case. One needs to proceed on a term by term basis matching the inner and outer solutions systematically and this will inform how the addtional terms behave. We will continue to the next order for both the inner and outer solutions. Now for the outer solution

$$y = Y_0 + \epsilon Y_1 + \dots,$$

and the problem for Y_1 is

$$2Y_1' - Y_1 = Y_0'' = \frac{1}{4} e^{\frac{x-1}{2}}, \quad Y_1(1) = 0.$$

Solving gives

$$Y_1 = \frac{(x-1)}{8} e^{\frac{x-1}{2}}.$$

For the inner problem, we have $x = \epsilon X$ and

$$y = y_0(X) + \epsilon y_1(X) + \dots .$$

The problem for y_1 is

$$\frac{d^2 y_1}{dX^2} + 2\frac{dy_1}{dX} = y_0 = e^{-\frac{1}{2}}(1 - e^{-2X}), \quad y_1(X = 0) = 0. \tag{5.6}$$

The solution of (5.6) gives

$$y_1 = A(1 - e^{-2X}) + \frac{1}{2}X(1 + e^{-2X})e^{-\frac{1}{2}},$$

where we have incorporated the boundary condition and A is an arbitrary constant to be determined from matching with the outer solution. The outer solution expanded for small x gives

$$y_{outer} = e^{\frac{x-1}{2}} + \epsilon\frac{1}{8}(x-1)e^{\frac{x-1}{2}} + \dots ,$$

$$\sim e^{-\frac{1}{2}}(1 + \frac{x}{2} + \dots) + \epsilon e^{-\frac{1}{2}}(\frac{(x-1)}{8}(1 + \frac{x}{2} + \dots)).$$

Written in terms of inner variables this is

$$y_{outer} \sim e^{-\frac{1}{2}} + \epsilon e^{-\frac{1}{2}}(\frac{X}{2} - \frac{1}{8}) + \dots .$$

The two term inner solution is

$$y_{inn} = e^{-\frac{1}{2}}(1 - e^{-2X}) + \epsilon[A(1 - e^{-2X}) + \frac{1}{2}X(e^{-2X} + 1)e^{-\frac{1}{2}}] + \dots ,$$

$$\sim e^{-\frac{1}{2}} + \epsilon(A + \frac{1}{2}Xe^{-\frac{1}{2}}) + \dots , \tag{5.7}$$

as $X \to \infty$. This has to match with the two term outer solution written in terms of inner variables, i.e.,

$$y_{outer} \sim e^{-\frac{1}{2}} + \epsilon e^{-\frac{1}{2}}(\frac{X}{2} - \frac{1}{8}) + \dots . \tag{5.8}$$

A match is only possible if $A = -\frac{1}{8}e^{-\frac{1}{2}}$. Thus

$$y_1 = -\frac{1}{8}e^{-\frac{1}{2}}(1 - e^{-2X}) + \frac{X}{2}e^{-\frac{1}{2}}(1 + e^{-2X}).$$

Video clip showing example of higher-order matching.

5.2 Uniform approximations

A uniform approximation to the solution valid in the whole region is defined by

$$y_{unif} = Y_{outer} + y_{inn} - y_{match}$$

where y_{match} is the approximation to $y(x)$ in the matching region.

For the above problem we had

$$Y_{outer} = e^{-\frac{1}{2}}e^{\frac{x}{2}} + \epsilon\frac{e^{-\frac{1}{2}}}{8}(x-1)e^{-\frac{x}{2}} + O(\epsilon^2).$$

$$y_{inn} = e^{-\frac{1}{2}}(1-e^{-\frac{2x}{\epsilon}}) + \epsilon[-\frac{1}{8}e^{-\frac{1}{2}}(1-e^{-\frac{2x}{\epsilon}}) + \frac{e^{-\frac{1}{2}}}{2}\frac{x}{\epsilon}(1+e^{-\frac{2x}{\epsilon}})] + \ldots.$$

The matching region is $X(= x/\epsilon) >> 1$ and $x << 1$, ie,

$$\epsilon << x << 1.$$

Thus a one-term uniform approximation is

$$y_{unif} = e^{-\frac{1}{2}}e^{\frac{x}{2}} + e^{-\frac{1}{2}}(1-e^{-\frac{2x}{\epsilon}}) - e^{-\frac{1}{2}}$$

ie

$$y_{unif} = e^{-\frac{1}{2}}[e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}}].$$

A two term uniform approximation is

$$y_{unif} = e^{-\frac{1}{2}}e^{\frac{x}{2}} + \epsilon\frac{e^{-\frac{1}{2}}}{8}(x-1)e^{-\frac{x}{2}}$$

$$+e^{-\frac{1}{2}}(1-e^{-\frac{2x}{\epsilon}}) + \epsilon[-\frac{1}{8}e^{-\frac{1}{2}}(1-e^{-\frac{2x}{\epsilon}}) + \frac{e^{-\frac{1}{2}}}{2}\frac{x}{\epsilon}(1+e^{-\frac{2x}{\epsilon}})]$$

$$-[e^{-\frac{1}{2}} + \epsilon(-\frac{1}{8} + \frac{x}{2\epsilon})e^{-\frac{1}{2}}].$$

ie

$$y_{unif} = e^{-\frac{1}{2}}(e^{\frac{x}{2}} - e^{-\frac{2x}{\epsilon}} + \frac{x}{2}e^{-\frac{2x}{\epsilon}}) + \epsilon(\frac{e^{-\frac{1}{2}}}{8}(x-1)e^{-\frac{x}{2}} + \frac{e^{-\frac{1}{2}}}{8}e^{-\frac{2x}{\epsilon}}).$$

Video clip for uniform approximations.

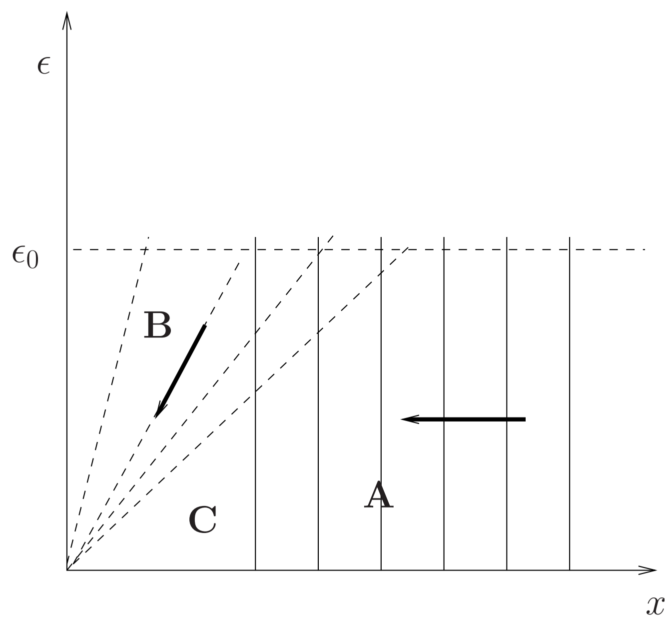

Figure 4: Outer solution represented by region **A** with $\epsilon \to 0$ x fixed, and inner solution by **B** with $\epsilon \to 0$ with $X = x/\epsilon$ fixed.

5.3 More on matching and intermediate variables

In the previous example we constructed an outer solution with x fixed and ϵ tending to zero, and an inner expansion with $X = x/\epsilon$ fixed and ϵ going to zero. Grapically the process may be represented as in fig. 4 with the region A representing the outer solution and region B the inner solution. The figure also shows an overlap region where the two solutions agree. However closer examination of the figure might suggest that there is a possibility of a region C not accessible by the inner or outer solutions. In reality the actual domains of validity of the two solutions may be larger than the above limiting process allows. The difficulty here is arises from the way the matching is done.

A different way to match the two solutions is to introduce an intermediate variable, say $x = \epsilon^\alpha \xi$ with (in the above example) $0 < \alpha < 1$. We have $X = x/\epsilon = \epsilon^{-1+\alpha}\xi$ and so as $\epsilon \to 0$ with ξ fixed gives $X \to \infty$ and $\epsilon \to 0$ with ξ fixed

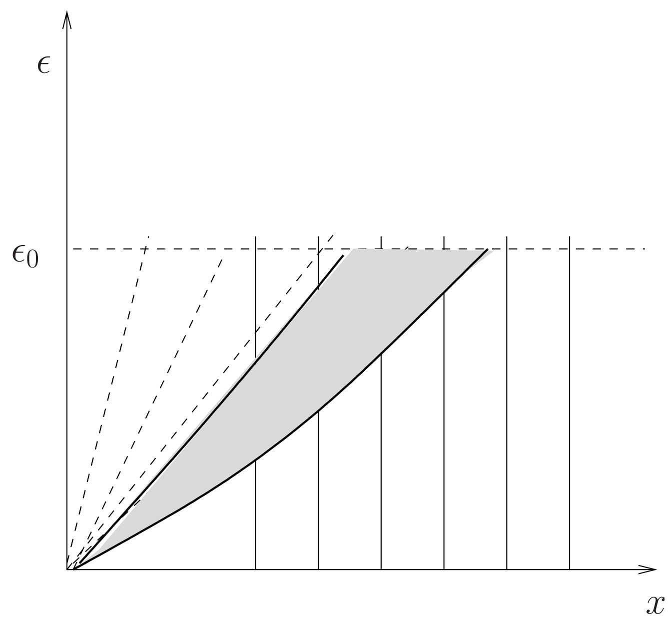

Figure 5: Overlap region (shaded) working in terms of intermediate variables $x = \epsilon^{\alpha}\xi$ and $X = \epsilon^{-1+\alpha}\xi$ with $0 < \alpha < 1$, and $\epsilon \to 0+$.

also gives $x \to 0$. Thus ξ is an *intermediate* variable and it is in this variable that we attempt to match the inner and outer solutions. The region defined by $\xi = O(1)$ is *an overlap region* for the two solutions, as shown schematically in fig. 5.

Video clip showing use of intermediate variables for previous example.

We will show how this works with another example in which the differential equation is nonlinear.

Example Consider

$$\epsilon y'' + y' + y^2 = 0, \quad y(0) = 0, y(1) = 1/2. \tag{5.9}$$

Suppose we look for an outer solution of the form

$$y = y_0 + \epsilon y_1 + \ldots .$$

Then from (5.9) we obtain

$$y_0' + y_0^2 = 0, \quad y_0'' + y_1' + 2y_0y_1 = 0. \tag{5.10}$$

The solution of the outer problem shows that

$$-\frac{y_0'}{y_0^2} = 1, \quad \frac{1}{y_0} = x + k,$$

and so

$$y_0 = \frac{1}{x+k}.$$

The boundary layer occurs at $x = 0$ (why?) and so we need to use the condition $y_0(1) = 1/2$ giving $k = 1$, and so

$$y_0 = \frac{1}{x+1}.$$

At next order

$$y_1' + 2y_0y_1 + y_0'' = 0, \quad y_1(1) = 0.$$

Substituting for $y_0 = 1/(x+1)$ gives

$$y_1' + \frac{2}{x+1}y_1 = \frac{-2}{(1+x)^3}.$$

Hence

$$((1+x)^2y_1)' = -\frac{2}{1+x},$$

$$(1+x)^2y_1 + k_1 = -2\log(x+1).$$

Applying the condition $y_1(1) = 0$ gives $k_1 = -2\log 2$ and thus

$$y_1 = \frac{2\log(\frac{2}{1+x})}{(1+x)^2}.$$

For the inner solution we need to seek a solution in terms of an inner variable say $x = \epsilon^n X$ and substitution in (5.9) shows that $n = 1$ for a distinguished limit. The inner solution may be expanded as

$$y = Y_0(X) + \epsilon Y_1(X) + \dots.$$

After substitution into (5.9) and using $x = \epsilon X$ we obtain

$$Y_0'' + Y_0' = 0, \quad Y_1'' + Y_1' + Y_0^2 = 0.$$

The boundary conditions are

$$Y_0(0) = 0, \quad Y_1(0) = 0.$$

Solving for Y_0 yields

$$Y_0 = A_0 + B_0 e^{-X}, \quad \text{and} \quad A_0 + B_0 = 0.$$

Thus

$$Y_0 = A_0(1 - e^{-X}).$$

To find A_0 we match with intermediate variables and put $x = \epsilon^{\alpha}\xi, X = \epsilon^{-1+\alpha}\xi$, and $0 < \alpha < 1$ with $\xi = O(1)$. The one term outer solution written in terms of ξ is

$$y = y_0(x) + \cdots \sim \frac{1}{1+\epsilon^{\alpha}\xi} \sim 1 - \epsilon^{\alpha}\xi + \ldots. \tag{5.11}$$

Similarly the outer solution in terms of ξ is

$$y = Y_0(X) + \cdots \sim A_0(1 - e^{-\epsilon^{-1+\alpha}\xi}) \sim A_0.$$

Thus matching with (5.11) shows that $A_0 = 1$ with error $O(\epsilon^{\alpha})$.

Before we match to second order we need to find Y_1 which satisfies

$$Y_1'' + Y_1' + Y_0^2 = 0, \quad Y_1(0) = 0.$$

Thus

$$Y_1'' + Y_1' = -(1 - e^{-X})^2.$$

Solving and applying the condition on $X = 0$ gives (check)

$$Y_1(X) = A_1(1 - e^{-X}) + \frac{1}{2}(1 - e^{-2X}) - X(1 + 2e^{-X}).$$

Next we write the outer and inner expansions in terms of the intermediate variables and do the matching. The outer expansion written in terms of ξ is

$$\begin{aligned} y_{out} &= \frac{1}{1+x} + \epsilon\frac{1}{(1+x)^2}2\log(\frac{2}{1+x}) + \ldots, \\ &= \frac{1}{1+\epsilon^{\alpha}\xi} + \epsilon\frac{1}{(1+\epsilon^{\alpha}\xi)^2}2\log(\frac{2}{1+\epsilon^{\alpha}\xi}) + \ldots, \\ &\sim 1 - \epsilon^{\alpha}\xi + \epsilon^{2\alpha}\xi^2 + \cdots + 2\log 2(\epsilon - 2\epsilon^{\alpha+1}\xi + O(\epsilon^{2\alpha})) - 2\epsilon(1 - 2\epsilon^{\alpha}\xi)(\epsilon^{\alpha}\xi - O(\epsilon^{2\alpha})). \end{aligned} \tag{5.12}$$

Next the inner solution written in terms of ξ is

$$\begin{aligned} y_{inn} &= (1 - e^{-X}) + \epsilon(A_1(1 - e^{-X}) + \frac{1}{2}(1 - e^{-2X}) - X(1 + 2e^{-X})) + \ldots, \\ &= (1 - e^{-\epsilon^{\alpha-1}\xi}) + \epsilon\left[A_1(1 - e^{-\epsilon^{\alpha-1}\xi}) + \frac{1}{2}(1 - e^{-2\epsilon^{\alpha-1}\xi}) - \epsilon^{\alpha-1}\xi(1 + 2e^{-\epsilon^{\alpha-1}\xi})\right] + \ldots, \\ &\sim 1 + \epsilon A_1 + \frac{\epsilon}{2} - \epsilon^{\alpha}\xi + \ldots \end{aligned} \tag{5.13}$$

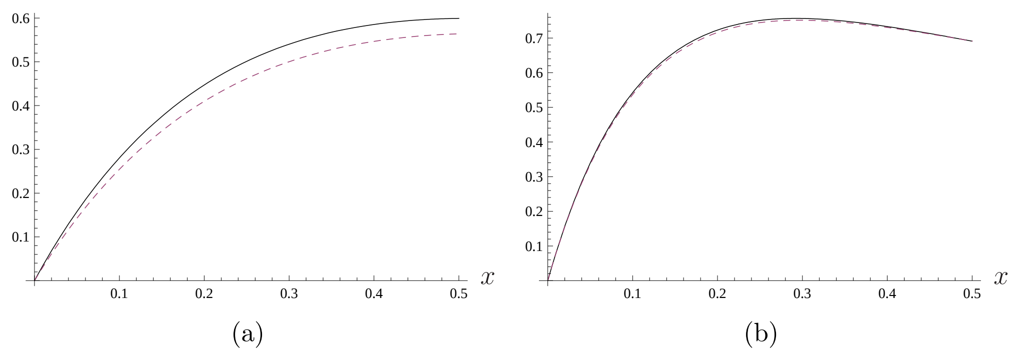

Figure 6: A comparison of the numerical solution of (5.9) (solid lines) with the composite solution given by (5.15) (dashed line) taking (a) $\epsilon = 0.2$, and (b) $\epsilon = 0.1$

In (5.12) if we keep terms to order ϵ and assuming that the terms $O(\epsilon^{2\alpha})$ are smaller than terms of $O(\epsilon)$ we require $0 < \alpha < 1/2$. This gives

$$y_{out} \sim 1 - \epsilon^{\alpha}\xi + \epsilon 2\log 2 + O(\epsilon^2, \epsilon^{1+\alpha}, \epsilon^{2\alpha}). \tag{5.14}$$

Comparing (5.14) and (5.13) we see that the terms of $O(\epsilon^{\alpha})$ match automatically and to match the $O(\epsilon)$ terms we require

$$\epsilon A_1 + \frac{\epsilon}{2} = 2\epsilon \log 2,$$

giving

$$A_1 = -\frac{1}{2} + 2\log 2.$$

At the next order of matching the terms of $O(\epsilon^{2\alpha})$ match automatically.

The composite solution to $O(\epsilon^2)$ is

$$y_{comp} = y_{out} + y_{inn} - y_{match}.$$

In the above example we find

$$y_{comp} = \frac{1}{x+1} + \frac{2\epsilon}{(x+1)^2}\log\frac{2}{x+1} + (1 - e^{-\frac{x}{\epsilon}}) +$$
$$\epsilon\left[(-\frac{1}{2} + 2\log 2)(1 - e^{-\frac{x}{\epsilon}}) + \frac{1}{2}(1 - e^{-\frac{2x}{\epsilon}}) - \frac{x}{\epsilon}(1 + 2e^{-\frac{x}{\epsilon}})\right]$$
$$-(1 + \epsilon(-\frac{1}{2} + 2\log 2 + \frac{1}{2} - \frac{x}{\epsilon})). \tag{5.15}$$

A comparison of the numerical solution of (5.9) with the composite solution is shown in Fig. (6) and shows excellent agreement for ϵ small.

5.4 Interior layers

Video clip covering an example of hidden boundary layer.

Consider

$$\epsilon y'' + a(x)y' + b(x)y = 0, \quad y(0) = A, y(1) = B. \tag{5.16}$$

The outer problem (set $\epsilon = 0$) is just

$$a(x)y' + b(x)y = 0.$$

Take $a(x) > 0$ and then

$$y' = -\frac{b(x)}{a(x)}y, \quad y = Ce^{-\int_{x_0}^{x} \frac{b(s)}{a(s)}\,ds}.$$

Again there are two boundary conditions to satisfy and so there must be a boundary layer, but where is the boundary located?

Video clip for example of boundary layer not at $x = 0$.

Suppose that we have a boundary layer at $x = \bar{x}$ of thickness $\gamma(\epsilon)$. We write

$$x = \bar{x} + \gamma(\epsilon)X, \quad \text{where} \quad X = O(1).$$

Then substituting into (5.16) with $y = Y$ gives

$$\frac{\epsilon}{\gamma^2}\frac{d^2Y}{dX^2} + \frac{a(\bar{x} + \gamma X)}{\gamma}\frac{dY}{dX} + b(\bar{x} + \gamma X)Y = 0.$$

Now expand a, b as

$$a(\bar{x} + \gamma X) = a(\bar{x}) + \gamma X a'(\bar{x}) + \dots, \quad b(\bar{x} + \gamma X) = b(\bar{x}) + \gamma X b'(\bar{x}) + \dots,$$

to get

$$\frac{\epsilon}{\gamma^2}\frac{d^2Y}{dX^2} + \frac{a(\bar{x})}{\gamma}\frac{dY}{dX} + b(\bar{x})Y + \dots = 0. \tag{5.17}$$

For $|\gamma| << 1$ the dominant terms are

$$\frac{\epsilon}{\gamma^2}\frac{d^2Y}{dX^2}, \quad \frac{a}{\gamma}\frac{dY}{dX}.$$

For a balance we require

$$\frac{\epsilon}{\gamma^2} \sim \frac{1}{\gamma} \qquad \Longrightarrow \gamma = O(\epsilon).$$

Hence set $\gamma = \epsilon$ ie $x = \bar{x} + \epsilon X$. From (5.17) the reduced inner equation is

$$\epsilon^{-1}\left[\frac{d^2Y}{dX^2} + a(\bar{x})\frac{dY}{dX}\right] + b(\bar{x})Y + \dots = 0. \tag{5.18}$$

The leading order inner problem is

$$\frac{d^2Y}{dX^2} + a(\bar{x})\frac{dY}{dX} = 0,$$

giving

$$Y = C_0 + C_1 e^{-a(\bar{x})X}.$$

Now we have assumed that $a(\bar{x}) > 0$. If $\bar{x} > 0$ we need to match as we go out of the boundary layer, ie we need limits $X \to \pm\infty$.

As $X \to \infty$ everything is ok, but as $X \to -\infty$ it suggests that C_1 must be zero to avoid exponential growth.

But $C_1 = 0$ implies no boundary layer. Hence $\bar{x} = 0$ and the boundary layer is at $x = 0$ if $a(x) > 0$.

Similarly if $a(x) < 0$ then we have a bounday layer at $x = 1$. If $a(x) = 0$ inside the region we have an internal boundary layer. The above analysis also breaks down.

5.5 Further Examples, interior layers

Consider

$$\epsilon y'' + xy' - (\epsilon^2 x^3 + 1)y = 0, \quad y(-1) = 1, y(1) = 2$$

and $-1 \le x \le 1, \quad 0 < \epsilon << 1$. The above discussion suggests an interior layer at $x = 0$.

Video clip for above section.

For the outer solution put

$$y = y_0 + \epsilon y_1 + \dots,$$

to get

$$xy_0' - y_0 = 0.$$

Thus

$$y_0 = Ax.$$

Here we have a new difficulty. Which boundary condition do we choose? We can show that there are no boundary layers near $x = \pm 1$. We write

$$y = A_{\pm} x$$

where the + stands for $x > 0$ and $-$ for $x < 0$.

From the boundary conditions it suggests that

$$A_+ = 2, \quad A_- = -1.$$

When x is small the $\epsilon y''$ term is not negligble, and hence we look for an interior layer at $x = 0$ and write

$$x = \gamma(\epsilon) X, \quad \gamma(\epsilon) << 1.$$

This gives with $y = Y$

$$\frac{\epsilon}{\gamma^2}\frac{d^2Y}{dX^2} + \frac{\gamma X}{\gamma}\frac{dY}{dX} - Y + \dots = 0.$$

For a dominant balance this suggests that

$$\frac{\epsilon}{\gamma^2} \sim O(1) \quad \Longrightarrow \gamma = O(\epsilon^{\frac{1}{2}}).$$

Hence set $x = \epsilon^{\frac{1}{2}} X$ and from the outer solution it suggests that we expand the inner solution as

$$y = \epsilon^{\frac{1}{2}} Y_0 + \epsilon Y_1 + \dots.$$

Video clip for interior layer problem, outer solution for above example.

Substituting into the equation gives

$$\epsilon\epsilon^{-1}\left(\epsilon^{\frac{1}{2}}\frac{d^2Y_0}{dX^2}+\dots\right)+\epsilon^{\frac{1}{2}}X\epsilon^{-\frac{1}{2}}\left(\epsilon^{\frac{1}{2}}\frac{dY_0}{dX}+\dots\right)-\epsilon^{\frac{1}{2}}Y_0+\dots=0.$$

Hence the leading order problem is

$$\frac{d^2Y_0}{dX^2}+X\frac{dY_0}{dX}-Y_0=0. \tag{5.19}$$

The boundary conditions suggest that we must match with the outer solution as $X\to\pm\infty$. This suggests that

$$Y_0\sim A_{\pm}X \quad \text{as} \quad X\to\pm\infty. \tag{5.20}$$

The equation (5.19) can be solved in terms of parabolic cylinder functions. If we put

$$Y_0=e^{-\frac{X^2}{4}}W_0$$

then W_0 satisfies

$$W_0''+(\frac{1}{2}-2-\frac{X^2}{4})W_0=0.$$

Note that two linearly independent solutions of the equation

$$W''+(\frac{1}{2}+\nu-\frac{X^2}{4})W=0,$$

are the parabolic cylinder functions $W=D_\nu(X)$ and $D_{-\nu-1}(iX)$.

In order to do the matching we require the behaviours of $D_\nu(x)$ for $|x|$ large. The properties of $D_\nu(z)$ are summarized below (see for example Abramovitz & Stegun[1]:

$$D_\nu(z)\sim z^\nu e^{-\frac{z^2}{4}}\sum_{n=0}^{\infty}(-1)^n a_n z^{-2n} \quad \text{as} \quad z\to\infty, \quad |\arg(z)|<\frac{3\pi}{4}. \tag{5.21}$$

[1] M. Abramovitz and I. A. Stegun *Handbook of Mathematical Function*, Dover. [web version also available]

$$D_\nu(z) \sim z^\nu e^{-\frac{z^2}{4}} \sum_{n=0}^{\infty} (-1)^n a_n z^{-2n} - \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)} e^{i\pi\nu} z^{-\nu-1} e^{\frac{z^2}{4}} \sum_{n=0}^{\infty} b_n z^{-2n}$$

$$\text{as} \quad z \to \infty, \quad \frac{\pi}{4} < \arg(z) < \frac{5\pi}{4}. \tag{5.22}$$

Here $a_0 = b_0 = 1$, and

$$a_n = \frac{\nu(\nu-1)\dots(\nu-2n+1)}{2^n n!} \quad b_n = \frac{(\nu+1)(\nu+2)\dots(\nu+n)}{2^n n!}.$$

Hence we can write the solution of (5.19) as

$$Y_0 = e^{-\frac{X^2}{4}}(CD_{-2}(X) + ED_1(iX)).$$

Now using

$$D_{-2}(X) \sim X^{-2} e^{-\frac{X^2}{4}}, \quad D_1(iX) \sim (iX) e^{\frac{X^2}{4}} \quad \text{as} \quad X \to \infty$$

and

$$D_{-2}(X) \sim X^{-2} e^{-\frac{X^2}{4}} - \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(2)} e^{2\pi i} X e^{\frac{X^2}{4}}, \quad \text{as} \quad X \to -\infty$$

$$D_1(iX) \sim (iX) e^{\frac{X^2}{4}} \quad \text{as} \quad X \to -\infty,$$

we find that

$$Y_0 \sim e^{-\frac{X^2}{4}} \left[\frac{C}{X^2} e^{-\frac{X^2}{4}} + E(iX) e^{\frac{X^2}{4}} \right] \quad X \to \infty,$$

ie

$$Y_0 \sim EiX \quad \text{as} \quad X \to \infty.$$

Hence

$$Ei = A_+.$$

Similarly

$$Y_0 \sim e^{-\frac{X^2}{4}} \left[-C\sqrt{(2\pi)} X e^{\frac{X^2}{4}} + E(iX) e^{\frac{X^2}{4}} \right] \quad X \to -\infty.$$

Hence

$$Y_0 \sim (-\sqrt{(2\pi)}C + iE)X + O(1) \quad X \to -\infty,$$

giving

$$-\sqrt{(2\pi)}C + iE = A_-.$$

Using the given values for $A_\pm$ leads to

$$C = \frac{3}{\sqrt{2\pi}}, \quad E = -2i,$$

and the inner solution as

$$Y_0 = \left(\frac{3}{\sqrt{2\pi}} D_{-2}(X) - 2iD_1(iX) \right) e^{-\frac{X^2}{4}}.$$

A uniform approximation can be calculated to give

$$y_{unif} = \epsilon^{\frac{1}{2}} \left(\frac{3}{\sqrt{2\pi}} D_{-2}(\frac{x}{\sqrt{\epsilon}}) - 2iD_1(\frac{ix}{\sqrt{\epsilon}}) \right) e^{-\frac{x^2}{4\epsilon}}.$$

A comparison of the uniform approxmation

$$y_{unif} = \epsilon^{\frac{1}{2}} \left(\frac{3}{\sqrt{2\pi}} D_{-2}(\frac{x}{\sqrt{\epsilon}}) - 2iD_1(\frac{ix}{\sqrt{\epsilon}}) \right) e^{-\frac{x^2}{4\epsilon}}.$$

with a numerical solution of the differential equation

$$\epsilon y'' + xy' - (\epsilon^2 x^3 + 1)y = 0, \quad y(-1) = 1, y(1) = 2$$

and $-1 \leq x \leq 1, \quad 0 < \epsilon << 1$, for $\epsilon = 0.05$ is shown in Fig. 7 below.

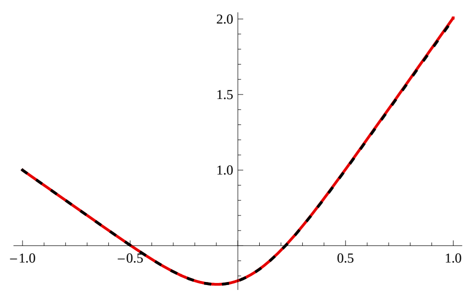

Figure 7: A comparison of the (exact) numerical solution to the full equation as compared with the uniform approximation (dashed line) for $\epsilon = 0.05$.

Video clip for interior layer problem, inner solution for above example.

5.6 Properties of parabolic cylinder functions

Consider the parabolic cylinder equations

$$y'' + (\nu + \frac{1}{2} - \frac{1}{4}z^2)y = 0.$$

All points except $z = \infty$ are ordinary points and one can readily obtain Taylor series solutions about $z = 0$. For $z \to \infty$ if we look for a solution of the form $y \sim e^{S(z)}$ we find that

$$y(x) \sim c_1 z^{-\nu-1} e^{\frac{z^2}{4}}, \quad \text{and} \quad y(x) \sim c_2 z^{\nu} e^{-\frac{z^2}{4}}, \quad z \to \infty.$$

The convention is to take $D_\nu(z)$ as the solution with the property that

$$y(z) \sim z^{\nu} e^{-\frac{z^2}{4}}, \quad z \to \infty.$$

Note that $D_\nu(-z)$ is also a solution and if we put $x = iz$ we find that the equation becomes

$$-\frac{d^2y}{dx^2} + (\nu + \frac{1}{2} + \frac{x^2}{4})y = 0,$$

ie

$$\frac{d^2y}{dx^2} + (-(\nu + \frac{1}{2}) - \frac{x^2}{4})y = 0,$$

or

$$\frac{d^2y}{dx^2} + (-(\nu + 1) + \frac{1}{2} - \frac{x^2}{4})y = 0.$$

Thus $y = D_{-\nu-1}(-iz)$ is also another (linearly independent) solution. A linear relationship must therefore exist between the three solutions and one can show that

$$D_\nu(z) = e^{i\nu z} D_\nu(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i(\nu+1)\frac{\pi}{2}} D_{-\nu-1}(-iz)$$

is valid for all z.

From the leading order asymptotic behaviour we see that the Stokes lines are given by

$$Re(z^2) = 0, \implies \arg(z) = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}.$$

The asymptotic behaviour for $D_\nu(z)$ as $z \to \infty$ can be obtained (see examples 2), and shows that

$$D_\nu(z) \sim z^{\nu} e^{-\frac{z^2}{4}} \sum_{n=0}^{\infty} (-1)^n a_n z^{-2n} \quad \text{as} \quad z \to \infty, \quad |\arg(z)| < \frac{3\pi}{4}. \tag{5.23}$$

Here $a_0 = 1$, and

$$a_n = \frac{\nu(\nu-1)\dots(\nu-2n+1)}{2^n n!}.$$

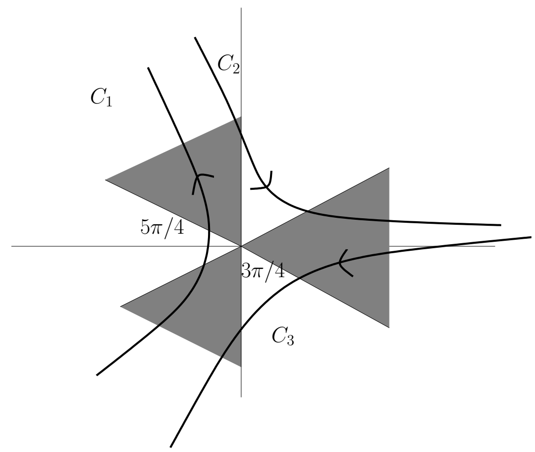

Figure 8: Stokes lines for parabolic cylinder function

Since $D_\nu(z)$ is subdominant in $|\arg(z)| < \pi/4$ the expression (5.23) is valid in the larger sector $|\arg(z)| < 3\pi/4$. [NB, as a rule of thumb this is generally true].

We can use the relation

$$D_\nu(z) = e^{i\nu z} D_\nu(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i(\nu+1)\frac{\pi}{2}} D_{-\nu-1}(-iz)$$

in conjunction with (5.23) to derive an expression valid for a larger sector.

Note that to use (5.23) with $D_\nu(-z)$ if we write $-z = |z|e^{-i\pi+i\theta}$ we require

$$-\frac{3\pi}{4} < -\pi + \theta < \frac{3\pi}{4}, \quad \Longrightarrow \frac{\pi}{4} < \theta < \frac{7\pi}{4}.$$

Similarly to use (5.23) with $D_{-\nu-1}(-iz)$ if we write $-iz = |z|e^{-i\frac{\pi}{2}+i\theta}$ we require

$$\frac{\pi}{4} < \theta < \frac{5\pi}{4}.$$

Hence for $\pi/4 < \arg(z) < 5\pi/4$ using (5.23) we find that

$$D_\nu(z) \sim z^\nu e^{-\frac{z^2}{4}} \sum_{n=0}^{\infty} (-1)^n a_n z^{-2n} - \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)} e^{i\pi\nu} z^{-\nu-1} e^{\frac{z^2}{4}} \sum_{n=0}^{\infty} b_n z^{-2n}$$

$$\text{as} \quad z \to \infty, \quad \frac{\pi}{4} < \arg(z) < \frac{5\pi}{4}. \tag{5.24}$$

Here $b_0 = 1$, and

$$b_n = \frac{(\nu+1)(\nu+2)\dots(\nu+n)}{2^n n!}.$$

6 The LG approximation, WKBJ Method

Boundary layer theory fails when we have a rapid variation in the solution throughout the region rather than locally at some location.

Example Consider

$$\epsilon y'' + by = 0, \quad y(0) = 0, \quad y(1) = 1,$$

where $b > 0$ and $0 < \epsilon << 1$. Note that the general solution is

$$y = \frac{\sin(x\sqrt{\frac{b}{\epsilon}})}{\sin(\sqrt{\frac{b}{\epsilon}})}.$$

The outer solution is just $y = 0$. For the inner solution, suppose we set

$$x = \bar{x} + \gamma(\epsilon)X, \quad \gamma << 1.$$

Then the equations gives

$$\frac{\epsilon}{\gamma^2}\frac{d^2y}{dX^2} + by = 0.$$

A dominant balance gives $\gamma = \epsilon^{\frac{1}{2}}$ and the resulting inner problem is

$$\frac{d^2y}{dX^2} + by = 0.$$

The solution gives

$$y = A\sin(\sqrt{b}X) + B\cos(\sqrt{b}X).$$

We can choose any $\bar{x}$ but note that for any choice of $\bar{x}$ the solution is not of boundary layer form and cannot be matched to the outer solution as $X \to \pm\infty$ because the inner solution oscillates.

Video clip for above example

Boundary layer theory fails for these types of singular perturbation problems in which we have wavelike behaviour (as opposed to dissipative or dispersive behaviour). The LG approximation or WKBJ theory is ideal for these classes of problems. The technique we describe below leads to an approximation which was obtained by [Liouville (1837)] and [Green (1837)]. In fact as noted earlier, [Carlini (1817)] had also used the same ideas.

The method is more commonly known as the WKBJ method after [Wentzel (1926)], [Kramers (1926)], [Brillouin (1926)], and [Jeffreys (1924)]. (Theoretical physicists call it the WKB method). However it is more correct it to call it the LG approximation which was used by Jeffreys, Wentzel, Kramers and Brillouin, to derive the connection formula in the presence of turning points (see later).

Consider

$$\epsilon y'' = Q(x)y, \quad Q(x) \neq 0. \tag{6.1}$$

The basic idea of the theory is that for $\epsilon \to 0$ we look for a solution to (6.1) of the form

$$y \sim A(x,\delta) e^{\frac{s(x,\delta)}{\delta}}, \quad \delta(\epsilon) \to 0$$

where $A(x,\delta), s(x,\delta)$ are slowly varying functions of x, but note the rapid variation of the solution because of the exponential factor. We can absorb the A into the exponential by writing

$$y = e^{\frac{S(x,\delta)}{\delta}}. \tag{6.2}$$

Substitution (6.2) into the equation (6.1) gives

$$\epsilon \left[\frac{S'^2}{\delta^2} + \frac{S''}{\delta} \right] - Q(x) = 0,$$

where primes denote differentiation with respect to x.

For a dominant balance we have $\delta = \epsilon^{\frac{1}{2}}$, and the equation for S reduces to

$$S'^2 - Q(x) = -\epsilon^{\frac{1}{2}} S''. \tag{6.3}$$

This suggests that we write

$$S = \sum_{n=0}^{\infty} \epsilon^{\frac{n}{2}} S_n, \quad \epsilon \to 0.$$

Substitution into (6.3) gives

$$(S_0' + \epsilon^{\frac{1}{2}} S_1' + \ldots)^2 - Q(x) = -\epsilon^{\frac{1}{2}} (S_0'' + \epsilon^{\frac{1}{2}} S_1'' + \ldots). \tag{6.4a}$$

Equating like powers of ϵ in (6.4a) to zero gives

$$(S_0')^2 = Q(x), \tag{6.4b}$$
$$2S_0' S_1' = -S_0'', \tag{6.4c}$$
$$2S_0' S_n' + \sum_{j=1}^{n-1} S_j' S_{n-j}' = -S_{n-1}'', \quad n \geq 2. \tag{6.4d}$$

We can solve (6.4b) to obtain

$$S_0 = \pm \int^x Q^{\frac{1}{2}}\, dx,$$

$$S_1' = -\frac{S_0''}{2S_0'} \quad \Longrightarrow S_1 = -\frac{1}{4}\log|Q|.$$

Hence the leading order behaviour of the solution can be written down as

$$y \sim |Q|^{-\frac{1}{4}} \left[C_1 \exp\left(\frac{1}{\epsilon^{\frac{1}{2}}}\int_a^x Q^{\frac{1}{2}}\right) + C_2 \exp\left(-\frac{1}{\epsilon^{\frac{1}{2}}}\int_a^x Q^{\frac{1}{2}}\right)\right], \tag{6.5}$$

where C_1, C_2, a are determined from the boundary conditions. This is the LG approximation to the solution. The approximation with just the leading order term S_0 gives what the physicists like to call the *geometrical optics* approximation. The approximation (6.5) is also referred to as the *physical optics* approximation.

Video clip for WKB method- general theory

Example Consider again

$$\epsilon y'' + by = 0, \quad y(0) = 0, y(1) = 1,$$

and $b > 0$. Here $Q(x) = -b$, and so

$$S_0 = \pm i\sqrt{b}x.$$

Hence using (6.5)

$$y \sim b^{-\frac{1}{4}}(C_1 e^{ix\sqrt{\frac{b}{\epsilon}}} + C_2 e^{-ix\sqrt{\frac{b}{\epsilon}}}),$$

or

$$y \sim A_1 \sin\left(\sqrt{\frac{b}{\epsilon}}x\right) + A_2 \cos\left(\sqrt{\frac{b}{\epsilon}}x\right).$$

Applying the boundary conditions gives the exact solution

$$y = \frac{\sin\left(\sqrt{\frac{b}{\epsilon}}x\right)}{\sin\left(\sqrt{\frac{b}{\epsilon}}\right)}.$$

Example Consider

$$\epsilon y'' - (1+x^2)^2 y = 0, \quad y(0) = 0, y'(0) = 1.$$

If we look for a solution

$$y \sim \exp\,(\frac{1}{\delta}\sum_0^\infty \delta^n S_n)$$

then again with $\delta = \epsilon^{\frac{1}{2}}$ we obtain

$$S_0'^2 = (1+x^2)^2, \quad S_0 = \pm(\frac{x^3}{3} + x).$$

Next

$$S_1 = -\frac{1}{4}\log(1+x^2)^2 = -\frac{1}{2}\log(1+x^2).$$

Thus

$$y \sim (1+x^2)^{-\frac{1}{2}}\left(C_1 \exp(\frac{\frac{x^3}{3}+x}{\epsilon^{\frac{1}{2}}}) + C_2 \exp(-\frac{\frac{x^3}{3}+x}{\epsilon^{\frac{1}{2}}})\right). \tag{6.6}$$

To find the constants we need to apply the boundary conditions. The condition $y(0) = 0$ leads to (after substituting $x = 0$ in (6.6)

$$0 = C_1 + C_2.$$

Now assuming that differentition of (6.6) is valid, we find that

$$y'(x) \sim \frac{(1+x^2)^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}}\left(C_1 \exp(\frac{(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}}) - C_2 \exp(\frac{-(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}})\right)$$

$$-x(1+x^2)^{-\frac{3}{2}}\left(C_1 \exp(\frac{(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}}) + C_2 \exp(\frac{-(\frac{x^3}{3}+x)}{\epsilon^{\frac{1}{2}}})\right).$$

Hence applying $y'(0) = 1$ gives

$$1 = \epsilon^{-\frac{1}{2}}(C_1 - C_2).$$

Solving for C_1, C_2 gives

$$C_1 = \frac{1}{2}\epsilon^{\frac{1}{2}}, \quad C_2 = -\frac{1}{2}\epsilon^{\frac{1}{2}}.$$

Hence a WKB approximation to the solution is

$$y \sim \epsilon^{\frac{1}{2}}(1+x^2)^{-\frac{1}{2}}\sinh\left(\frac{\frac{x^3}{3}+x}{\epsilon^{\frac{1}{2}}}\right), \quad \epsilon \to 0.$$

The WKB method can also be used for certain eigenvalue problems.

Example Consider

$$y'' + \lambda p(x) y = 0, \quad y(0) = 0, \quad y(\pi) = 0. \tag{6.7}$$

This equation has nontrivial solutions only for certain discrete values of λ say $(\lambda_1, \lambda_2, \dots)$. We can obtain an approximation to the eigenvalues and eigenfunction for large λ.

Look for an asymptotic solution in WKB form as

$$y \sim \exp(\lambda^{\frac{1}{2}} \sum_{n=0}^{\infty} \lambda^{-n/2} S_n(x)).$$

Substitution into the equation (6.7) gives

$$\lambda(S_0' + \lambda^{-\frac{1}{2}} S_1' + \dots)^2 + \lambda^{\frac{1}{2}}(S_0'' + \lambda^{-\frac{1}{2}} S_1'' + \dots) + \lambda p(x) = 0.$$

Solving for S_0, S_1 gives

$$S_0 = \pm i \int^x (p(x))^{\frac{1}{2}}\, dx, \quad S_1 = -\frac{1}{4} \log |p(x)|.$$

Hence

$$y \sim |p|^{-\frac{1}{4}} \left[C_1 \sin(\lambda^{\frac{1}{2}} \int_0^x (p(x))^{\frac{1}{2}}\, dx) + C_2 \cos(\lambda^{\frac{1}{2}} \int_0^x (p(x))^{\frac{1}{2}}\, dx) \right].$$

The boundary conditions in (6.7)imply

$$C_2 = 0,$$

and

$$\sin(\lambda^{\frac{1}{2}} \int_0^{\pi} (p(x))^{\frac{1}{2}}\, dx) = 0.$$

Hence

$$\lambda^{\frac{1}{2}} = \frac{\pm n\pi}{\int_0^{\pi} (p(x))^{\frac{1}{4}}\, dx} \quad .$$

Thus

$$\lambda \sim \lambda_n = \frac{n^2 \pi^2}{[\int_0^{\pi} (p(x))^{\frac{1}{4}}\, dx]^2}, \quad n >> 1,$$

and approximate solution to (6.7) is

$$y \sim |p|^{-\frac{1}{4}} C_n \sin(\lambda_n^{\frac{1}{2}} \int_0^x (p(x))^{\frac{1}{2}}\, dx).$$

6.1 Additional notes

Implicit in the use of the WKB (LG) method ie

$$y \sim \exp(\sum_{n=0}^{\infty} \delta^{n-1} S_n(x))$$

is that the series

$$\sum_{n=0}^{\infty} \delta^{n-1} S_n(x), \quad \text{as} \quad \delta \to 0$$

is an asymptotic series, uniformly valid for all x throughout the interval. This requires that

$$\delta^n S_{n+1}(x) = o(\delta^{n-1} S_n(x)), \quad n = 1, 2, \ldots,$$

holds uniformly in x.

Since we take the exponential of the above series, for the WKB (LG) approximation to be a good approximation, if we truncate the series at $n = M - 1$ say, then we should have

$$\delta^M S_{M+1}(x) = o(1) \quad \delta \to 0$$

since

$$\exp(\delta^M S_{M+1}(x)) = 1 + O(\delta^M S_{M+1}(x)), \quad \text{as} \quad \delta \to 0.$$

6.2 Turning points and connection formulae

So far in

$$\epsilon y'' - Q(x, \epsilon) y = 0$$

we have taken $Q(x, \epsilon) > 0$ in the interval.

We will now consider

$$\epsilon y'' - Q(x) y = 0, \quad a < x < b, \quad Q(x_0) = 0, \quad Q'(x_0) > 0, \quad a < x_0 < b. \tag{6.8}$$

We will assume that there is only one zero in the $a < x < b$. A WKB approximation to the equation (6.8) is

$$y \sim C|Q(x)|^{-\frac{1}{4}} \exp\left(\pm \frac{1}{\epsilon^{\frac{1}{2}}} \int^x (Q(s))^{\frac{1}{2}}\, ds.\right).$$

Thus for $x > x_0$ we write

$$y \sim |Q(x)|^{-\frac{1}{4}} \left[A_1 \exp\left(\frac{1}{\epsilon^{\frac{1}{2}}} \int^x (Q(s))^{\frac{1}{2}}\, ds\right) + A_2 \exp\left(-\frac{1}{\epsilon^{\frac{1}{2}}} \int^x (Q(s))^{\frac{1}{2}}\, ds.\right)\right], \tag{6.9}$$

and for $x < x_0$ we have

$$y \sim |Q|^{-\frac{1}{4}} \left[B_1 \cos(\frac{1}{\epsilon^{\frac{1}{2}}} \int^x |Q(s)|^{\frac{1}{2}}\, ds) + B_2 \sin(\frac{1}{\epsilon^{\frac{1}{2}}} \int^x |Q(s)|^{\frac{1}{2}}\, ds)\right]. \tag{6.10}$$

The above approximation fails near $x = x_0$, where we have

$$Q(x) \sim (x - x_0)Q'(x_0) + \dots. \qquad (6.11)$$

If we put $x = x_0 + \epsilon^\gamma X$ and substitute into the differential equation (6.8) and use (6.11) we obtain

$$\epsilon\epsilon^{-2\gamma}\frac{d^2y}{dX^2} - (\epsilon^\gamma XQ'(x_0)y + \dots) = 0.$$

For a dominant balance we require

$$\epsilon^{1-2\gamma} \sim \epsilon^\gamma, \quad \Longrightarrow \gamma = \frac{1}{3}.$$

The dominant equation in this region reduces to Airy's equation

$$\frac{d^2y}{dX^2} - Xcy = 0, \quad c = Q'(x_0) > 0.$$

This has the solution

$$y_{inn} = D_1\mathrm{Ai}(c^{\frac{1}{3}}X) + D_2\mathrm{Bi}(c^{\frac{1}{3}}X), \qquad (6.12)$$

which is the inner solution. We need to match this with the outer solution (6.9.6.10) as $X \to \pm\infty$ or $x \to x_0 \pm$. Now

$$\mathrm{Ai}(X) \sim \frac{1}{2\sqrt{\pi}}X^{-\frac{1}{4}}e^{-\frac{2}{3}X^{\frac{3}{2}}}, \quad X \to \infty,$$

$$\mathrm{Bi}(X) \sim \frac{1}{\sqrt{\pi}}X^{-\frac{1}{4}}e^{\frac{2}{3}X^{\frac{3}{2}}}, \quad X \to \infty.$$

Thus for $X \to +\infty$ from (6.12)

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}X^{-\frac{1}{4}}c^{-\frac{1}{12}}\left(\frac{D_1}{2}e^{-\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}} + D_2e^{\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}}\right).$$

Also

$$\mathrm{Ai}(X) \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}\sin(\frac{2}{3}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) \quad X \to -\infty,$$

$$\mathrm{Bi}(X) \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}\cos(\frac{2}{3}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) \quad X \to -\infty.$$

Hence from (6.12) for $X \to -\infty$

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}c^{-\frac{1}{12}}\left[D_1\sin(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) + D_2\cos(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4})\right]$$

Now if we take the lower limit in (6.9) to be equal to x_0 (this is not necessary but it simplifies the expressions) then

$$\int_{x_0}^{x} (Q(s))^{\frac{1}{2}}\,ds = \int_0^{x-x_0} [Q(x_0+T)]^{\frac{1}{2}}\,dT,$$

$$\sim \int_0^{x-x_0} \left[cT + \frac{Q''(x_0)}{2}T^2 + \dots\right]^{\frac{1}{2}} dT,$$

$$\sim \int_0^{x-x_0} c^{\frac{1}{2}}T^{\frac{1}{2}}\left[1 + \frac{Q''(x_0)}{4c}T + \dots\right] dT.$$

Hence

$$\int_{x_0}^{x} (Q(s))^{\frac{1}{2}}\,ds \sim \frac{2}{3}c^{\frac{1}{2}}(x-x_0)^{\frac{3}{2}}$$

Hence as $x \to x_0+$ the outer solution behaves as

$$y_{out}^{+} \sim [c(x-x_0)]^{-\frac{1}{4}}\left[A_1\exp(\frac{1}{\epsilon^{\frac{1}{2}}}\frac{2}{3}c^{\frac{1}{2}}(x-x_0)^{\frac{3}{2}}) + A_2\exp(-\frac{1}{\epsilon^{\frac{1}{2}}}\frac{2}{3}c^{\frac{1}{2}}(x-x_0)^{\frac{3}{2}})\right],$$

ie

$$y_{out}^{+} \sim c^{-\frac{1}{4}}\epsilon^{\frac{-1}{12}}X^{-\frac{1}{4}}\left[A_1\exp(\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}) + A_2\exp(-\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}})\right].$$

Also

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}X^{-\frac{1}{4}}c^{-\frac{1}{12}}\left(\frac{D_1}{2}e^{-\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}} + D_2e^{\frac{2}{3}c^{\frac{1}{2}}X^{\frac{3}{2}}}\right). \tag{6.13}$$

To match with the inner solution (6.13) as $X >> 1$ we must have

$$\frac{D_1}{2\sqrt{\pi}}c^{-\frac{1}{12}} = A_2c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}, \quad \frac{D_2}{\sqrt{\pi}}c^{-\frac{1}{12}} = A_1c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}.$$

Similarly as $x \to 0-$ we have

$$\int_{x_0}^{x} (-Q(s))^{\frac{1}{2}}\,ds \sim -\frac{2}{3}c^{\frac{1}{2}}(x_0-x)^{\frac{3}{2}}.$$

Thus

$$y_{out}^{-} \sim c^{-\frac{1}{4}}(x_0-x)^{-\frac{1}{4}}\left[B_1\cos(\frac{2}{3}c^{\frac{1}{2}}(x_0-x)^{\frac{3}{2}}) - B_2\sin(\frac{2}{3}c^{\frac{1}{2}}(x_0-x)^{\frac{3}{2}})\right],$$

$$y_{out}^{-} \sim c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}|X|^{-\frac{1}{4}}\left[B_1\cos(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}}) - B_2\sin(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}})\right]. \tag{6.14}$$

And

$$y_{inn} \sim \frac{1}{\sqrt{\pi}}(-X)^{-\frac{1}{4}}c^{-\frac{1}{12}}\left[D_1\sin(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4}) + D_2\cos(\frac{2}{3}c^{\frac{1}{2}}(-X)^{\frac{3}{2}} + \frac{\pi}{4})\right] \tag{6.15}$$

To match (6.14, 6.15) we must have

$$c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}B_1 = \frac{c^{-\frac{1}{12}}}{\sqrt{2\pi}}(D_1 + D_2) = (\frac{A_1}{\sqrt{2}} + A_2\sqrt{2})c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}},$$

$$-c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}B_2 = \frac{c^{-\frac{1}{12}}}{\sqrt{2\pi}}(D_1 - D_2) = -(\frac{A_1}{\sqrt{2}} - A_2\sqrt{2})c^{-\frac{1}{4}}\epsilon^{-\frac{1}{12}}.$$

Hence solving for B_1, B_2 gives

$$B_1 = \frac{A_1}{\sqrt{2}} + A_2\sqrt{2},$$

$$B_2 = \frac{A_1}{\sqrt{2}} - A_2\sqrt{2}.$$

Summary: For $x > x_0$

$$y \sim |Q(x)|^{-\frac{1}{4}}\left[A_1 \exp\left(\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}(Q(s))^{\frac{1}{2}}\,ds\right)\right.$$

$$\left. +A_2 \exp\left(-\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}(Q(s))^{\frac{1}{2}}\,ds.\right)\right], \qquad (6.16a)$$

and for $x < x_0$ we have

$$y \sim |Q|^{-\frac{1}{4}}\left[A_1 \sin(\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}|Q(s)|^{\frac{1}{2}}\,ds + \frac{\pi}{4})\right]$$

$$\left. +2A_2 \cos(\frac{1}{\epsilon^{\frac{1}{2}}}\int_{x_0}^{x}|Q(s)|^{\frac{1}{2}}\,ds + \frac{\pi}{4})\right]. \qquad (6.16b)$$

For $(x - x_0) << 1$

$$y \sim \sqrt{\pi}c^{-\frac{1}{6}}\epsilon^{-\frac{1}{12}}\left[2A_2\mathrm{Ai}(c^{\frac{1}{3}}\epsilon^{-\frac{1}{3}}(x - x_0)) + A_1\mathrm{Bi}(c^{\frac{1}{3}}\epsilon^{-\frac{1}{3}}(x - x_0))\right].$$

The formulae (6.16) are known as the connection formulae. The constants A_1, A_2 are determined by the boundary conditions. ** CHECK video**

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7 Generalised Functions

Consider the the following function.

$$\delta_\epsilon(x) = \begin{cases} 0, & x < 0, \\ \epsilon^{-1}, & 0 < x < \epsilon, \\ 0, & x > \epsilon. \end{cases}$$

If $f(x)$ is continuous in an interval which includes the origin and $(0, \epsilon)$ then

$$\int_{-\infty}^{\infty} \delta_\epsilon(x) f(x)\, dx = \epsilon^{-1} \int_0^\epsilon f(x)\, dx.$$

From the mean value theorem

$$\int_0^\epsilon f(x)\, dx = \epsilon f(\epsilon\xi), \quad 0 \le \xi \le 1,$$

and therefore

$$\int_{-\infty}^{\infty} \delta_\epsilon(x) f(x)\, dx = f(\epsilon\xi), \quad 0 \le \xi \le 1.$$

If we let $\epsilon \to 0$ we obtain

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0),$$

for all functions $f(x)$ continuous in a neighbourhood including the origin. The function $\delta(x)$ is the limit of the $\delta_\epsilon(x)$ as $\epsilon \to 0$ is called the *delta function*, and is an example of a *generalised function.* Note that

$$\delta(x) = \begin{cases} 0, & x < 0 \\ 0, & x > 0. \end{cases},$$

and is undefined at $x = 0$. It is not an ordinary function and also not integrable in the usual sense.

The concept of a delta function dates back to the time of [Kirchoff (1882)] and [Heaviside (1893)]. The physicist Paul Dirac, after whom the function is named, in the 1920's popularised the concept of the delta function in quantum mechanics, see [Dirac (1947)], but from a mathematical viewpoint there were many shortcomings. The theory of generalised functions dates back to the work of Sobolev (1936) and [Schwartz (1950)], [Schwartz (1951)]. A popular and readable text (*An introduction to Fourier Analysis and generalised functions*) was produced by [Lighthill (1958)] based on the theory of [Mikunsinksi (1948)] and [Temple (1953)], [Temple (1955)]. The brief introduction below follows Lighthill's book, but see also [Jones (1982)] (*Generalised Functions*) which does things in a more formal setting. If you want a very formal treatment with linear functionals and measure theory, the book by [Vladimirov (2002)] *Methods of the theory of generalised functions* is highly recommended.

We first need to introduce the idea of what Lighthill calls *good functions* and *fairly good functions*.

Definition *We say that* $f(x) \in \mathcal{C}^m(a,b)$ *if* $f(x)$ *and its first* m *derivatives are continuous in the interval* (a,b).

$f(x) \in \mathcal{C}^\infty(R)$ *is the class of infinitely smooth function in* R.

Example The function $e^{-x^2} \in \mathcal{C}^\infty(R)$.

Definition *A function is said to belong to* $\mathcal{G}$ *if* $f(x) \in \mathcal{C}^\infty(R)$ *and*

$$\lim_{|x|\to\infty} |x^m \frac{d^k}{dx^k} f(x)| = 0$$

for every k *and for every integer* $m \geq 0$.

The space $\mathcal{G}$ *is the space of* good functions *in the sense of Lighthill. The space* $\mathcal{G}$ *is also called the* Schwartz space.

Example $e^{-x^2} \in \mathcal{G}$.

Definition *A function is said to belong to* $\mathcal{N}$ *if* $f(x) \in \mathcal{C}^\infty(R)$ *and if there exists some* N *such that*

$$\lim_{|x|\to\infty} |x^{-N} \frac{d^k}{dx^k} f(x)| = 0$$

for every $k \geq 0$.

The space $\mathcal{N}$ *is the space of* fairly good functions *in the sense of Lighthill.*

Example $x^p \in \mathcal{N}$. Any polynomial expression belongs to $\mathcal{N}$.

The following properties are straightforward to demonstrate.

- $f(x) \in \mathcal{G} \implies f'(x) \in \mathcal{G}$.
- $f(x), g(x) \in \mathcal{G} \implies f(x) + g(x) \in \mathcal{G}$.
- $f(x) \in \mathcal{G}, g(x) \in \mathcal{N} \implies f(x)g(x) \in \mathcal{G}$.

Definition *A sequence* $\{\phi_n(x)\}_{n=1}^\infty$, *and* $\phi_n(x) \in \mathcal{G}$ *is called a* regular sequence in $\mathcal{G}$ *if for any* $f(x) \in \mathcal{G}$ *the limit*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x) f(x)\, dx$$

exists.

Two regular sequences $\{\phi_n(x)\}_{n=1}^{\infty}, \{\psi_n(x)\}_{n=1}^{\infty}$, *are* equivalent sequences in $\mathcal{G}$ *if*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\phi_n(x)f(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}\psi_n(x)f(x)\,dx.$$

Example $e^{-x^2/n^2}, e^{-x^4/n^2}$ are equivalent sequences in $\mathcal{G}$.

Definition *Each equivalent class of regular sequences in* $\mathcal{G}$ *defines a* generalised function.

Definition *The sequence* $\delta_n(x) = \sqrt{\frac{n}{\pi}}e^{-nx^2}$ *defines the function* $\delta(x)$ *such that*

$$\int_{-\infty}^{\infty}\delta(x)f(x)\,dx = f(0)$$

for $f(x)\in\mathcal{G}$.

Proof We have to prove that the limit of the sequence

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)f(x)\,dx = f(0).$$

Now

$$\int_{-\infty}^{\infty}\delta_n(x)\,dx = \int_{-\infty}^{\infty}\sqrt{\frac{n}{\pi}}e^{-nx^2}\,dx = 1.$$

Also

$$|f(x)-f(0)| = |\int_0^x f'(s)\,ds| \le M|x|,$$

since $f(x)\in\mathcal{G}$ and is bounded. Thus

$$\left|\int_{-\infty}^{\infty}\delta_n(x)f(x)\,dx - f(0)\right| = \left|\int_{-\infty}^{\infty}\delta_n(x)(f(x)-f(0))\,dx\right|$$

$$\le M\int_{-\infty}^{\infty}|x|\delta_n(x)\,dx = M\int_{-\infty}^{\infty}|x|\sqrt{\frac{n}{\pi}}e^{-nx^2}\,dx$$

$$\le 2M\sqrt{\frac{n}{\pi}}\int_0^{\infty}xe^{-nx^2}\,dx = \frac{M}{\sqrt{n\pi}}[-e^{nx^2}]_0^{\infty} = \frac{M}{\sqrt{n\pi}}.$$

Hence taking the limit as $n\to\infty$ proves the result.

7.1 Derivatives of generalised functions

Suppose that $\{\phi_n(x)\}_{n=0}^{\infty}$ is a regular sequence in $\mathcal{G}$ then since $\phi_n'(x) \in \mathcal{G}$ we have after integrating by parts

$$\int_{-\infty}^{\infty} \phi_n'(x) f(x)\, dx = -\int_{-\infty}^{\infty} \phi_n(x) f'(x)\, dx,$$

for every $f(x) \in \mathcal{G}$. Letting $n \to \infty$ we see that $\{\phi_n'(x)\}_{n=0}^{\infty}$ is also a regular sequence. We denote the generalised function defined by this sequence as $\phi'(x)$ and we see that

$$\int_{-\infty}^{\infty} \phi'(x) f(x)\, dx = -\int_{-\infty}^{\infty} \phi(x) f'(x)\, dx,$$

We can continue in this way and we see that generalised functions possess derivatives to all orders and in an obvious notation

$$\int_{-\infty}^{\infty} \frac{d^k \phi(x)}{dx^k} f(x)\, dx = (-1)^k \int_{-\infty}^{\infty} \phi(x) \frac{d^k}{dx^k} f(x)\, dx.$$

Example

$$\int_{-\infty}^{\infty} \frac{d^k \delta(x)}{dx^k} f(x)\, dx = (-1)^k f^{(k)}(0).$$

** CHECK video**

Video clip on introduction to generalised functions.

Suppose $\{\phi_n(x)\}_{n=0}^{\infty}$ is a regular sequence in $\mathcal{G}$. Then

$$\int_{-\infty}^{\infty} \phi_n(ax+b) F(x)\, dx = |a|^{-1} \int_{-\infty}^{\infty} \phi_n(x) F(\frac{x-b}{a})\, dx.$$

Hence for $F(x) \in \mathcal{G}$

$$\int_{-\infty}^{\infty} \phi(ax+b) F(x)\, dx = |a|^{-1} \int_{-\infty}^{\infty} \phi(x) F(\frac{x-b}{a})\, dx, \quad a \neq 0.$$

Example

$$\int_{-\infty}^{\infty} \delta(ax-b)F(x)\,dx = |a|^{-1}\int_{-\infty}^{\infty} \delta(x)F(\frac{x+b}{a})\,dx = |a|^{-1}F(\frac{b}{a}), \quad a \neq 0.$$

Definition *We say that* $f(x) \in L_p(R)$ *if* $\int_{-\infty}^{\infty} |f(x)|^p\,dx$ *exists.*

Thus for example $L_1(R)$ is the space of absolutely integrable functions. ** CHECK video**

Video clip discussing properties of generalised functions.

Definition *The function* $f(x) \in K_p(R)$ *if for some* $N \geq 0$

$$\int_{-\infty}^{\infty} \frac{|f(x)|^p}{(1+x^2)^N}\,dx < \infty.$$

Example Consider the Heaviside function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

Clearly $H(x)$ is not absolutely integrable but $H(x) \in K_1(R)$, with $N = 1$.

Example $f(x) = H(x)x^3 \in K_1(R)$ with $N = 3$.

Theorem *Suppose* $f(x) \in K_1(R)$*. Then it is possible to construct a regular sequence* $\{\phi_n(x)\}_{n=0}^{\infty}$ *in* $\mathcal{G}$ *which defines a generalised function* $\phi(x)$ *such that*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} \phi_n(x)F(x)\,dx = \int_{-\infty}^{\infty} \phi(x)F(x)\,dx = \int_{-\infty}^{\infty} f(x)F(x)\,dx, \tag{7.1}$$

for any $F(x) \in \mathcal{G}$.

Proof For a proof see Jones, section 3.2.

The main point of this theorem is that it allows one to define a whole range of generalised functions for which the integral on the right hand side of (7.1) exists. The integral exists in the normal sense.

Example Consider the Heaviside function introduced earlier. This satisfies the condition of the theorem. Hence we can consider $H(x)$ as a generalised function and using an earlier result for the derivatives of generalised functions, we have

$$\int_{-\infty}^{\infty} H'(x)F(x)\,dx = -\int_{-\infty}^{\infty} H(x)F'(x)\,dx$$

for any $F(x) \in \mathcal{G}$. Now

$$\int_{-\infty}^{\infty} H(x)F'(x)\,dx = \int_{0}^{\infty} F'(x)\,dx = [F(x)]_0^{\infty} = -F(0).$$

Hence we see that

$$\int_{-\infty}^{\infty} H'(x)F(x)\,dx = F(0)$$

and thus

$$H'(x) = \delta(x).$$

Example Consider the function sgn(x) defined by

$$\mathrm{sgn(x)} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

This satisfies the condition of the theorem with $N = 1$. Hence

$$\int_{-\infty}^{\infty} \frac{d\mathrm{sgn(x)}}{\mathrm{dx}} F(x)\,dx = -\int_{-\infty}^{\infty} \mathrm{sgn(x)F'(x)\,dx},$$

$$= \int_{-\infty}^{0} F'(x)\,dx - \int_{0}^{\infty} F'(x)\,dx = 2F(0).$$

Hence we have

$$\frac{d\mathrm{sgn(x)}}{\mathrm{dx}} = 2\delta(x).$$

Consider the function $|x|^{\alpha}$. Now

$$\int_{-\infty}^{\infty} (1 + x^2)^{-N} |x|^{\alpha}\,dx$$

is convergent only if $\alpha > -1$ and if we take $2N > 1 + \alpha$. Thus we can define a generalised function $|x|^{\alpha}$ if $\alpha > -1$. Now

$$\frac{d}{dx}\log|x| = |x|^{-1}\mathrm{sgn(x)},$$

and so

$$\frac{d}{dx}|x|^{\alpha} = \alpha|x|^{\alpha-1}\text{sgn(x)},$$

provided also $\alpha > 0$. ** CHECK video**

Video clip discussing extensions to include functions in $K_p(R)$.

We make use of a result which states that if $f(x)$ is an ordinary function and both $f'(x)$ and $f(x)$ belong to $\mathcal{K}_1(R)$ then the derivative of the generalised function formed by $f(x)$ is the generalised function formed by $f'(x)$.

This can be used to define generalised functions such as $|x|^{\alpha}$ for non-integral $\alpha < 0$. For all α and α not equal to a negative integer, we can define the generalised functions

$$|x|^{\alpha} = \frac{1}{(\alpha+1)(\alpha+2)\dots(\alpha+n)}\frac{d^n}{dx^n}[|x|^{\alpha+n}(\text{sgn}(x))^n],$$

$$|x|^{\alpha}\text{sgn}(x) = \frac{1}{(\alpha+1)(\alpha+2)\dots(\alpha+n)}\frac{d^n}{dx^n}[|x|^{\alpha+n}(\text{sgn}(x))^{n+1}],$$

$$|x|^{\alpha}H(x) = \frac{1}{(\alpha+1)(\alpha+2)\dots(\alpha+n)}\frac{d^n}{dx^n}[x^{\alpha+n}H(x))],$$

where n is a positive integer such that $n + \Re(\alpha) > -1$. For completeness, the generalised function x^{-1} is defined by

$$x^{-1} = \frac{d}{dx}[\log|x|],$$

and if m is a positive integer

$$x^{-m} = \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}(\log|x|)^{m-1}.$$

7.2 Application to singular integrals

We will show one use of the above results for handling singular integrals. But first we need the following result.

Suppose $f(x)$ is a continuous function with a derivative $f'(x)$ both belonging to $\mathcal{K}_1$. Then

$$\frac{d}{dx}[f(x)H(x-a)] = \frac{df}{dx}H(x-a) + f(a)\delta(x-a).$$

Proof

With the given conditions $f(x)H(x-a)$ and the derivative $\frac{d}{dx}[f(x)H(x-a)]$ define generalised functions. Hence for any good function $\phi(x)$ we have

$$\int_{-\infty}^{\infty} \frac{d}{dx}[f(x)H(x-a)]\phi(x)\,dx = -\int_{-\infty}^{\infty} f(x)H(x-a)\phi'(x)\,dx$$

$$= -\int_{a}^{\infty} f(x)\phi'(x)\,dx.$$

Integrating by parts we obtain

$$-\int_{-\infty}^{\infty} f(x)H(x-a)\phi'(x)\,dx = -[f(x)\phi(x)]_a^{\infty} + \int_a^{\infty} f'(x)\phi(x)\,dx$$

$$= \int_{-\infty}^{\infty} [f'(x)H(x-a) + \delta(x-a)f(a)]\phi(x)\,dx.$$

Hence

$$\frac{d}{dx}[f(x)H(x-a)] = \frac{df}{dx}H(x-a) + f(a)\delta(x-a).$$

Consider the integral

$$\int_a^b \frac{1}{x}\phi(x)\,dx$$

where $\phi(x)$ is a continuous differentiable function of x and $\phi(0) \neq 0$ and $a < 0 < b$. In the normal sense the integral does not exist since

$$\left(\lim_{\epsilon_1 \to 0-} \int_a^{\epsilon_1} + \lim_{\epsilon_2 \to 0+} \int_{\epsilon_2}^{b}\right) \frac{\phi(x)}{x}\,dx$$

does not exist if ϵ_1, ϵ_2 go to zero independently.

However the *Cauchy principal value* of the integral is defined by

$$⨍_a^b \frac{\phi(x)}{x}\,dx = \lim_{\epsilon \to 0}\left(\int_a^{-\epsilon} + \int_{\epsilon}^{b}\right)\frac{\phi(x)}{x}\,dx$$

$$= \phi(b) \log|b| - \phi(a) \log|a| - \int_a^b \phi'(x) \log|x| \, dx.$$

Example

$$⨍_{-1}^{2} \frac{1}{x} \, dx = \log(2).$$

Let us see how we can tackle the integral using generalised functions. Consider

$$\int_a^b \frac{\phi(x)}{x} \, dx = \int_{-\infty}^{\infty} [H(x-a)x^{-1} - H(x-b)x^{-1}]\phi(x) \, dx.$$

Now

$$[(H(x-a) - H(x-b)) \log|x|]' =$$
$$[H(x-a) \log(|x|/|a|) - H(x-b) \log(|x|/|b|)$$
$$+H(x-a) \log|a| - H(x-b) \log|b|]'$$
$$= \{x^{-1}H(x-a) - x^{-1}H(x-b)\} + \delta(x-a) \log|a| - \delta(x-b) \log|b|.$$

Hence

$$\int_{-\infty}^{\infty} [H(x-a)x^{-1} - H(x-b)x^{-1}]\phi(x) \, dx =$$
$$\int_{-\infty}^{\infty} \{(H(x-a) - H(x-b)) \log|x|]'\phi(x) \, dx$$
$$+ \int_{-\infty}^{\infty} (\delta(x-b) \log|b| - \delta(x-a) \log|a|)\phi(x) \, dx$$
$$= \phi(b) \log|b| - \phi(a) \log|a| - \int_{-\infty}^{\infty} \{(H(x-a) - H(x-b)) \log|x|\}\phi'(x) \, dx$$

which is the same as the Cauchy principal value interpretation.

Consider next singular integrals of the form

$$\int_0^b x^\beta \phi(x) \, dx$$

where $0 < b$ and β is not a negative integer. We can write the integral as

$$\int_{-\infty}^{\infty} (x^\beta H(x) - x^\beta H(x-b))\phi(x) \, dx.$$

Next note that

$$[x^{\beta+n} H(x-b)]' = [(x^{\beta+n} - b^{\beta+n})H(x-b) + b^{\beta+n} H(x-b)]'.$$

$$= (\beta + n)x^{\beta+n-1}H(x-b) + b^{\beta+n}\delta(x-b).$$

Continue differentiating like this to obtain

$$[x^{\beta+n}H(x-b)]^{(n)} =$$

$$(\beta+n)(\beta+n-1)...(\beta+1)x^{\beta}H(x-b) + b^{\beta+n}\delta^{(n-1)}(x-b)+$$

$$+(\beta+n)b^{\beta+n-1}\delta^{(n-2)}(x-b) + \cdots + (\beta+n)...(\beta+2)b^{\beta+1}\delta(x-b).$$

Hence the integral can be written as

$$\int_{-\infty}^{\infty}\{x^{\beta}H(x) - x^{\beta}H(x-b)\}\phi(x)\,dx =$$

$$\int_{-\infty}^{\infty}\left[\frac{(x^{\beta+n}H(x) - x^{\beta+n}H(x-b))^{(n)}}{(\beta+n)(\beta+n-1)...(\beta+1)}\right.$$

$$\left.+\frac{b^{\beta+n}\delta^{(n-1)}(x-b)}{(\beta+n)...(\beta+1)} + \cdots + \frac{b^{\beta+1}\delta(x-b)}{\beta+1}\right]\phi(x)\,dx.$$

Finally we obtain

$$\int_0^b x^{\beta}\phi(x)\,dx =$$

$$\frac{(-1)^n}{(\beta+1)(\beta+2)...(\beta+n)}\int_{-\infty}^{\infty}x^{\beta+1}(H(x)-H(x-b))\phi^{(n)}(x)\,dx$$

$$+\frac{b^{\beta+1}}{(\beta+1)}\phi(b) - \frac{b^{\beta+2}\phi'(b)}{(\beta+1)(\beta+2)} + \cdots + \frac{(-1)^{n-1}b^{\beta+n}\phi^{(n-1)}(b)}{(\beta+1)(\beta+2)...(\beta+n)}.$$

The above intepretation agrees with the *Hadamard finite part* of the integral $\int_0^b x^{\beta}\phi(x)\,dx$.

Example Consider

$$\int_0^{*b} x^{-3/2}f(x)\,dx = \lim_{\epsilon\to 0}\int_{\epsilon}^{b} x^{-3/2}f(x)\,dx.$$

[The notation $\int^*$ is sometimes used to denote a singular integral and that we need to interpret in the Hadamard finite part sense]. Now

$$\int_{\epsilon}^{b} x^{-3/2}f(x)\,dx. = [-2x^{-1/2}f(x)]_{\epsilon}^{b} + \int_{\epsilon}^{b} 2x^{-1/2}f(x)\,dx$$

$$= [-2b^{-1/2}f(b)] + 2\epsilon^{-1/2}f(\epsilon) + \int_{\epsilon}^{b} 2x^{-1/2}f'(x)\,dx$$

The Hadamard finite part of the integral is defined by ignoring the $\epsilon^{-1/2} f(\epsilon)$ term and taking the limit as $\epsilon \to 0$ giving

$$\int_0^{*b} x^{-3/2} f(x)\, dx = [-2b^{-1/2} f(b)] + \int_\epsilon^b 2x^{-1/2} f'(x)\, dx.$$

Example Consider

$$\int_0^1 \frac{x^{-5/2}}{1+x}\, dx$$

We can write

$$x^{-\frac{5}{2}}(H(x) - H(x-1)) =$$
$$\frac{1}{(-\frac{5}{2}+1)(-\frac{5}{2}+2)} \frac{d^2}{dx^2}\left[x^{-\frac{1}{2}}(H(x) - H(x-1))\right]$$
$$+\frac{\delta(x-1)}{(-\frac{5}{2}+1)} + \frac{\delta'(x-1)}{(-\frac{5}{2}+1)(-\frac{5}{2}+2)}.$$

Hence with $\phi(x) = 1/(x+1)$ we have

$$\int_0^1 \frac{x^{-\frac{5}{2}}}{1+x}\, dx = \int_{-\infty}^{\infty} \frac{4}{3}\frac{d^2}{dx^2}\left[x^{-\frac{1}{2}}(H(x) - H(x-1))\right]\phi(x)\, dx$$
$$+\int_{-\infty}^{\infty}(-\frac{2}{3}\delta(x-1) + \frac{4}{3}\delta'(x-1))\phi(x)\, dx,$$
$$= \frac{4}{3}\int_{-\infty}^{\infty} x^{-\frac{1}{2}}(H(x) - H(x-1))\phi''(x)\, dx - \frac{2}{3}\phi(1) - \frac{4}{3}\phi'(1)$$
$$= \frac{4}{3}\int_0^1 \frac{2}{x^{\frac{1}{2}}(x+1)^3}\, dx = \frac{\pi}{2} + \frac{4}{3}$$

We will investigate Fourier transforms properties of generalised functions after we have discussed Fourier transforms.

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