A SHORT GEOMETRIC PROOF THAT HAUSDORFF LIMITS ARE DEFINABLE IN ANY O-MINIMAL STRUCTURE

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Abstract. The aim of this note is to give yet another proof of the following theorem: given an arbitrary o-minimal structure on the ordered field of real numbers $\mathbb{R}$ and any definable family $\mathcal{A}$ of definable nonempty compact subsets of $\mathbb{R}^n$, then the closure of $\mathcal{A}$ in the sense of the Hausdorff metric (or, equivalently, in the Vietoris topology) is a definable family. In particular, any limit in the sense of the Hausdorff metric of a convergent sequence of subsets of a definable family is definable in the same o-minimal structure. The original proofs by Bröcker [B], Marker and Steinhorn [MS], Pillay [P] (see also van den Dries [vdD2]) were based on model theory. Lion and Speissegger [LS] gave a geometric proof of the theorem. Our proof below is based on the idea of Lipschitz cell decompositions.

Introduction.

Let $\mathcal{K}_n$ denote the space of all non-empty compact subsets of the n-dimensional euclidean space $\mathbb{R}^n$ with the *Hausdorff metric*

$$d_H(A,B) := \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\},$$

where $d(x,B) := \inf_{y\in B} |x-y|$ and $A, B \in \mathcal{K}_n$.[1]

Fix an o-minimal structure on the ordered field of real numbers $\mathbb{R}$ (see [vdD1] or [C], for fundamental definitions and results concerning o-minimal structures). The aim of this note is to give a short geometric proof of the following theorem (compare with Theorem 2 in [LS]).

Main Theorem. *Let T be a definable bounded subset of $\mathbb{R}^k$ and let A be a definable bounded subset of $\mathbb{R}^n \times T$ such that all the* fibers $A_t := \{x \in \mathbb{R}^n : (x,t) \in A\}$ $(t \in T)$ *are non-empty compact subsets of $\mathbb{R}^n$.*

Then there exists a definable bounded subset S of $\mathbb{R}^k$ and a definable Lipschitz bijection $\varphi : S \longrightarrow T$ such that the mapping $S \ni s \longmapsto A_{\varphi(s)} \in \mathcal{K}_n$ is Lipschitz. Consequently, it extends in a unique way by continuity to $\overline{S}$.

Before stating corollaries to Main Theorem we adopt the following definition.

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[1] The Hausdorff metric defines the Vietoris topology in $\mathcal{K}_n$.

Definition 1. *A subset* $\mathcal{A}$ *of* $\mathcal{K}_n$ *will be called* **definable** *if there exists a definable subset* T *of some space* $\mathbb{R}^k$ *and a definable subset* A *of* $\mathbb{R}^n \times T$ *such that* $\mathcal{A} = \{A_t : t \in T\}$.

Corollary 1. *The closure* $\overline{\mathcal{A}}$ *of any definable subset* $\mathcal{A}$ *of* $\mathcal{K}_n$ *is definable. More precisely, if* $\mathcal{A} = \{A_t : t \in T\}$, *where* T *is a definable bounded subset of* $\mathbb{R}^k$ *and* A *is a definable subset of* $\mathbb{R}^n \times T$, *then there exists a definable bounded subset* S *of* $\mathbb{R}^k$ *and a definable Lipschitz bijection* $\varphi : S \longrightarrow T$ *such that, setting* $B := \{(x, s) : s \in S, x \in A_{\varphi(s)}\}$, *we have the following*

(i) $B_s = A_{\varphi(s)}$, *for each* $s \in S$.

(ii) $\overline{\mathcal{A}} = \{B_s : s \in \tilde{S}\}$, *where* $\tilde{S} = \{s \in \overline{S} : B_s \in \mathcal{K}_n\}$ *is open in* $\overline{S}$ *and equal to* $\pi(\overline{\{(A_{\varphi(s)}, s) \in \mathcal{K}_n \times S : s \in S\}})$, *where* $\pi : \mathcal{K}_n \times \overline{S} \ni (K, s) \longmapsto s \in \overline{S}$.

(iii) *The mapping* $\tilde{S} \ni s \longmapsto B_s \in \mathcal{K}_n$ *is continuous.*

Proof. If $\mathcal{A}$ is bounded in $\mathcal{K}^n$, then A is bounded and the corollary follows immediately from Main Theorem with $\tilde{S} = \overline{S}$ and the Lipschitz mapping in (iii). The case where $\mathcal{A}$ is not bounded in $\mathcal{K}_n$ reduces to the previous one via the semialgebraic homeomorphism

$$\mathbb{R}^n \ni x \longmapsto \frac{x}{\sqrt{1 + |x|^2}} \in \mathbb{B}_n := \{u \ni \mathbb{R}^n : |u| < 1\}.$$

Corollary 2. *If* $\mathcal{A} = \{A_t : t \in T\}$ *is any definable subset of* $\mathcal{K}_n$ *and* $C = \lim_{\nu \to \infty} A_{t_\nu}$ *is the limit of any convergent in* $\mathcal{K}_n$ *sequence* $A_{t_\nu} \in \mathcal{A}$ $(\nu \in \mathbb{N})$, *where* $\lim_{\nu \to \infty} t_\nu = t_* \,(\in \overline{T})$, *then* C *is definable and there exists a definable arc* $\gamma : (0, 1] \longrightarrow T$ *such that* $\lim_{\tau \to 0} \gamma(\tau) = t_*$ *and* $C = \lim_{\tau \to 0} A_{\gamma(\tau)}$.

Proof. By Corollary 1, $\varphi(s_\nu) = t_\nu$, where $s_\nu \in S$ $(\nu \in \mathbb{N})$. By passing to a subsequence, we can assume that $\lim_{\nu \to \infty} s_\nu = s_* (\in \overline{S})$ and, by (ii) and (iii), $s_* \in \tilde{S}$ and $B_{s_*} = C$. Take any definable arc $\lambda : (0, 1] \longrightarrow S$ such that $\lim_{\tau \to 0} = s_*$. Then $\gamma = \varphi \circ \lambda$ is a desired arc.

The original proofs of the above results were based on model theory (cf. Bröcker [B], Marker and Steinhorn [MS], Pillay [P], van den Dries [vdD2]). Lion and Speissegger [LS] use blowing up in jet spaces to give purely geometric proof. Our proof beneath is based on Lipschitz cell decompositions.

1. Reduction of the problem by a decomposition into Lipschitz cells.

The first ingredient of our proof is a version with a parameter of the Kurdyka-Parusiński Theorem on a decomposition of a definable subset into Lipschitz cells (cf. [K] and [Par]). Before stating it we first recall a definition of a Lipschitz cell in $\mathbb{R}^n$, which is by induction on n. Let M be a positive real number.

Definition 2. *In* $\mathbb{R}$ *Lipschitz cells are exactly singletons and open intervals.*

Assume that $n > 1$. *A subset* C *of* $\mathbb{R}^n$ *is a* **Lipschitz cell** *with constant* M *if either*

(1) $C = \{(y, x_n) : y \in D, x_n = f(y)\}$, *where* $y = (x_1, ..., x_{n-1})$, D *is a Lipschitz cell with constant* M *in* $\mathbb{R}^{n-1}$ *and* $f : D \longrightarrow \mathbb{R}$ *is a definable Lipschitz function with constant* M,

or

(2) $C = \{(y, x_n) : y \in D, f_1(y) < x_n < f_2(y)\}$, *where* y *and* D *are as in the case* (1), $f_i : D \longrightarrow \overline{\mathbb{R}}$, *for* $i \in \{1, 2\}$, $f_1(y) < f_2(y)$, *for each* $y \in D$, *and finally each of* f_i *is either a definable Lipschitz function with values in* $\mathbb{R}$ *and constant* M, *or identically* $-\infty$, *or identically* $+\infty$.

Theorem 1. *Let* A *be a definable subset of* $\mathbb{R}^n \times \mathbb{R}^k$.

Then there exists a finite decomposition $A = B_1 \cup ... \cup B_p$ *into definable subsets, there exist linear orthogonal automorphisms* $\varphi_j : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $(j = 1, ..., p)$ *and a positive constant* M_n *depending only on* n *such that, for each* $t \in \mathbb{R}^k$ *and each* $j \in \{1, ..., p\}$, *if* $B_{jt} \neq \emptyset$, *then* $\varphi_j(B_{jt})$ *is a Lipschitz cell in* $\mathbb{R}^n$ *with constant* M_n. *Consequently,* $A_t = B_{1t} \cup ... \cup B_{pt}$, *for each* $t \in \mathbb{R}^k$.

Proof. See Proposition 1.4 in [KP] or [K-C].

We will be often using the following.

Observation 1

If $T = \cup_{j=1}^{s} T_j$ is any finite partition into definable subsets and if we prove Main Theorem for every T_j separately obtaining an appropriate Lipschitz bijection $\varphi_j : S_j \longrightarrow T_j$, then we can obviously assume that the distances of different S_j's are positive; hence, gluing φ_j's together gives the desired φ for T.

It follows from Theorem 1 and Observation 1, that it suffices to prove Main Theorem for a family of closures of bounded definable Lipschitz cells in $\mathbb{R}^n$. We will see in a moment that in fact it suffices to prove the following proposition.

Proposition 1. *Let* $F : K \times T \ni (x, t) \longmapsto F_t(x) \in \mathbb{R}$ *be a bounded definable function, where* T *is a definable bounded subset of* $\mathbb{R}^k$ *and* K *is a compact* m*-dimensional interval in* $\mathbb{R}^m$*; i.e.,* $K = [a_1, b_1] \times ... \times [a_m, b_m]$, *where* $a_1 < b_1$, ..., $a_m < b_m$. *Assume that all the functions* F_t *are Lipschitz with a common Lipschitz constant* M.

Then there exists a definable bounded subset S *of* $\mathbb{R}^k$ *and a definable Lipschitz bijection* $\varphi : S \longrightarrow T$ *such that* $F \circ (id_K \times \varphi) : K \times S \ni (x, s) \longmapsto F(x, \varphi(s))$ *is Lipschitz.*

To derive the Main Theorem from Proposition 1, assume that $\{C_t\}$, $(t \in T)$ is a definable family of definable Lipschitz cells in $\mathbb{R}^n$ with a common Lipschitz constant M and such that all C_t are contained in one common bounded subset. We can distinguish the following two cases.

Case I

$$C_t = \{(y, x_n) : y \in D_t,\ x_n = f_t(y)\} \quad (t \in T)$$

where $y = (x_1, ..., x_{n-1})$, $\{D_t\}$ $(t \in T)$ is a definable family of definable subsets of $\mathbb{R}^{n-1}$, all contained in a common compact m-dimensional interval K, and where $f_t : D_t \to \mathbb{R}$ are commonly bounded Lipschitz functions with a common Lipschitz constant M. By the formula

$$F_t(z) = sup\{f_t(y) - M|z - y| : y \in D_t\}$$

we extend our family to a family of Lipschitz functions $F_t : K \to \mathbb{R}$ with constant M (compare [M] or [W; footnote on p. 63])[2]. Since

$$\overline{C}_t = \{(z, x_n) : z \in \overline{D}_t,\ x_n = F_t(z)\},$$

Proposition 1 together with the inductive argument on the dimension n finishes the proof in this case.

Case II

$$C_t = \{(y, x_n) : y \in D_t,\ f_t(y) < x_n < g_t(y)\} \quad (t \in T)$$

where y, D_t and f_t are as in Case 1 and $g_t : D_t \to \mathbb{R}$ are commonly bounded Lipschitz with a common Lipschitz constant M and such that $f_t(y) < g_t(y)$, for each $y \in D_t$.

Applying the argument of Case 1 to the both families $\{f_t\}$ and $\{g_t\}$, we finish the proof in this case.

2. Main lemma to prove Proposition 1.

Main lemma to prove Proposition 1 is the following.

Lemma 1. [3]
Let U be an open definable subset of $\mathbb{R}^k$ and let

$$f : K \times U \ni (x, u) \longmapsto f(x, u) \in \mathbb{R}$$

be a definable function, where $x = (x_1, ..., x_m)$ and $u = (u_1, ..., u_k)$. Let E be a closed, nowhere dense definable subset of $K \times U$ such that f is of class $\mathcal{C}^1$ outside E. Assume that

$$\left|\frac{\partial f}{\partial x_i}\right| \le M \quad (i = 1, ..., m) \quad on \quad \mathbb{R}^m \times U \setminus E.$$

Then there exists a closed nowhere dense definable subset Σ of U such that
(*) for each $u \in U \setminus \Sigma$, the subset E_u of $\mathbb{R}^m$ is nowhere dense, and
(**) for each $u \in U \setminus \Sigma$, the function

$$K \setminus E_u \ni x \longmapsto \frac{\partial f}{\partial u_1}(x, u) \in \mathbb{R}$$

is bounded.

To prove Lemma 1 we will use the following particular case of the $\mathcal{C}^1$-extension theorem [Paw; Proposition 2].

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[2]The last formula can be used to get a Lipschitz extension to the whole space $\mathbb{R}^{n-1}$.
[3]Compare [V; Proposition 5.5]

Theorem 2. *Let Δ be a definable open subset of $\mathbb{R}^k$ and let*

$$S = \{(u,v) \in \Delta \times \mathbb{R} : \varphi(u) < v < \psi(u)\},$$

where $\varphi : \Delta \to \mathbb{R}$ and $\psi : \Delta \to \mathbb{R}$ are definable $\mathcal{C}^1$-functions such that $\varphi(u) < \psi(u)$, for each $u \in \Delta$.

Let $g : S \longrightarrow \mathbb{R}$ be a definable $\mathcal{C}^1$-function such that the derivative $\dfrac{\partial g}{\partial v}$ is bounded.

Then there is a closed nowhere dense definable subset Z of (the graph of)[4] φ such that g extends to a $\mathcal{C}^1$-function

$$g : S \cup (\varphi \setminus Z) \longrightarrow \mathbb{R}$$

to $S \cup (\varphi \setminus Z)$ as a $\mathcal{C}^1$-submanifold of $\mathbb{R}^{k+1}$ with boundary $\varphi \setminus Z$.

Proof of Lemma 1. Of course there exists a closed nowhere dense Σ for which the condition (1) is satisfied, so without any loss of generality we can assume it to be satisfied for each $u \in U$.

Suppose that Lemma 1 is not true. Then there exists a nonempty open subset W of U such that for each $u \in W$ there exists $a \in E_u$ such that

$$\limsup_{\substack{x \to a \\ x \notin E_u}} \left| \frac{\partial f}{\partial u_1}(x,u) \right| = +\infty.$$

By definable choice we find a definable mapping $h = (h_1, ..., h_m) : W \to \mathbb{R}^m$ such that for each $u \in W,\ h(u) \in E_u$ and

$$\limsup_{\substack{x \to h(u) \\ x \notin E_u}} \left| \frac{\partial f}{\partial u_1}(x,u) \right| = +\infty.$$

Shrinking W we can assume that h is of class $\mathcal{C}^1$. By a version with a parameter of Curve Selecting Lemma, there exists a definable mapping $\alpha : (0,1) \times W \to \mathbb{R}^m$ such that every $\alpha_u : (0,1) \to \mathbb{R}^m \setminus E_u$ is a $\mathcal{C}^1$-arc lying outside E_u, $\lim_{t \to 0} \alpha_u(t) = h(u)$ and

$$\lim_{t \to 0} \frac{\partial f}{\partial u_1}(\alpha_u(t), u) = \pm\infty \tag{1}$$

Shrinking perhaps W and replacing the parameter t by $t' = \rho t$, with ρ small positive, we can assume that $\alpha = (\alpha_1, ..., \alpha_m)$ is $\mathcal{C}^1$ on $(0,1) \times W$, there exists $j \in \{1, ..., m\}$ such that

$$\left| \frac{\partial \alpha_j}{\partial t}(t,u) \right| \geq \left| \frac{\partial \alpha_i}{\partial t}(t,u) \right|, \quad \text{for each} \quad (t,u) \in (0,1) \times W,\ i \in \{1, ..., m\}, \tag{2}$$

and that the sign of $\dfrac{\partial \alpha_j}{\partial t}$ is constant, say positive, on $(0,1) \times W$. Of course we can assume that $j = 1$.

[4]We identify here a function with its graph.

Let us introduce the new variable $v = \alpha_1(t,u)$ in the place of t. Then $t = \beta_1(v,u)$, where $v \in (h_1(u), \omega(u))$, with a $\mathcal{C}^1$-function $\omega > h_1$. Let us set $\beta_i(v,u) = \alpha_i(\beta_1(v,u),u)$, for each $i \in \{2,...,m\}$ and $v \in (h_1(u), \omega(u))$.

By (2) we have $\left|\frac{\partial \beta_i}{\partial v}(v,u)\right| \leq 1$, for each $v \in (h_1(u), \omega(u))$ and $i \in \{2,...,m\}$. By Theorem 2, shrinking perhaps W, we can assume that all β_i $i \in \{2,...,m\}$ extend to $\mathcal{C}^1$-functions on $\{(v,u) : u \in W, v \in [h_1(u), \omega(u))\}$. In particular we can assume that the derivatives $\frac{\partial \beta_i}{\partial u_1}(v,u)$ are bounded, when v is sufficiently near $h_1(u)$.

In the new variable v (1) reads as follows

$$\lim_{v \to h_1(u)} \frac{\partial f}{\partial u_1}(v, \beta_2(v,u), ..., \beta_m(v,u), u) = \pm\infty. \tag{3}$$

Consider now the following function

$$g(v,u) = f(v, \beta_2(v,u), ..., \beta_m(v,u), u), \quad \text{for } u \in W \text{ and } v \in (h_1(u), \omega(u)).$$

It is easy to see that $\left|\frac{\partial g}{\partial v}(v,u)\right| \leq Mm$. On the other hand we have $\frac{\partial g}{\partial u_1}(v,u) =$

$$\sum_{i=2}^{m} \frac{\partial f}{\partial x_i}(v, \beta_2(v,u), ..., \beta_m(v,u), u) \frac{\partial \beta_i}{\partial u_1} + \frac{\partial f}{\partial u_1}(v, \beta_2(v,u), ..., \beta_m(v,u), u)$$

which - in virtue of (3) - tends to $\pm\infty$, for each $u \in W$, when v tends to $h_1(u)$. This is a contradiction with Theorem 2. The proof of Lemma 1 is complete.

3. Proof of Proposition 1.

We will argue by induction on $k = \dim T$. We will be often making use of the following

Observation 2

By the induction hypothesis that Proposition 1 is true if the subset of parameters is of dimension $< k$ and by Observation 1, we can remove from T any definable subset of dimension $< k$.

By Observations 1 and 2 together with Theorem 1 (where $T = \{t_o\}$), we can assume that $T = U$ is an open bounded subset of $\mathbb{R}^k$. We will first proof the following special case of Proposition 1:

Proposition 2. *Proposition 1 is true under the extra assumptions that* $T = U$ *is an open bounded subset of* $\mathbb{R}^k$ *and* $\left|\frac{\partial F}{\partial u_1}(x,u)\right| \leq M$, *for each* $u \in U$ *and* $x \in K \setminus E_u$.

To prove Proposition 2, we will use the following lemma.

Lemma 2. *Let* $c, d \in \mathbb{R},\ c < d$ *and let*

$$h : K \times (c,d) \ni (x,v) \longmapsto h(x,v) = h_v(x) \in \mathbb{R}$$

be a definable function such that all functions h_v *are Lipschitz with a common constant. Then there exists a finite partition* $c_0 = c < c_1 < \ldots < c_s = b$ *such that* h *is continuous on each of the sets* $K \times (c_{i-1}, c_i)$ $(i = 1,\ldots,s)$.

Proof of Lemma 2. Let Z denote the set of points of $K \times (c,d)$, at which h is not continuous. Suppose that $(x_o, v_o) \in Z$. Put

$$\sigma = \lim_{v \to {v_o}^+} h(x_o, v)\,(\in \overline{\mathbb{R}}) \quad \text{and} \quad \rho = \lim_{v \to {v_o}^-} h(x_o, v)\,(\in \overline{\mathbb{R}}).$$

Since h_v are equicontinuous, by the double limit theorem,

$$\sigma = \lim_{\substack{x \to x_o \\ v \to {v_o}^+}} h(x,v) \quad \text{and} \quad \rho = \lim_{\substack{x \to x_o \\ v \to {v_o}^-}} h(x,v).$$

Since $x \longmapsto h(x, v_o)$ is continuous and $(x_o, v_o) \in Z$, there must be $\sigma \neq h(x_o, v_o)$ or $\rho \neq h(x_o, v_o)$. Suppose, for example, that $\sigma < h(x_o, v_o)$. Take any σ' such that $\sigma < \sigma' < h(x_o, v_o)$. There exists a neighborhood U_o of x_o and $\epsilon > 0$ such that $h(x,v) < \sigma'$, whenever $x \in U_o$ and $v \in (v_o, v_o + \epsilon)$. Since $x \longmapsto h(x, v_o)$ is continuous, after perhaps shrinking U_o, we can assume that $\sigma' < h(x, v_o)$, for each $x \in U_o$. Consequently, for each $x \in U_o$,

$$\lim_{v \to {v_o}^+} h(x,v) \le \sigma' < h(x, v_o), \quad \text{thus} \quad U_o \subset Z_{v_o}.$$

Proof of Proposition 2. Let $\pi : U \ni u = (u_1, \ldots, u_k) \longmapsto \tilde{u} = (u_2, \ldots, u_k) \in \pi(U)$ be the natural projection. By Observations 1-2, without any loss of generality, we can assume that all the fibers of π are connected; i.e.

$$\pi^{-1}(\{\tilde{u}\}) = (c(\tilde{u}), d(\tilde{u})) \times \{\tilde{u}\}, \quad \text{for each} \quad \tilde{u} \in \pi(U).$$

By Lemma 2, we can also assume that $F_{\tilde{u}}$ is continuous on $K \times (c(\tilde{u}), d(\tilde{u}))$, for each $\tilde{u} \in \pi(U)$. Since $F_{\tilde{u}}$ is $\mathcal{C}^1$ outside a definable nowhere dense subset, by the Mean Value Theorem, $F_{\tilde{u}}$ is Lipschitz on $K \times (c(\tilde{u}), d(\tilde{u}))$, with a constant independent of $\tilde{u}$. As in Section 1, we now extend all $F_{\tilde{u}}$ to Lipschitz functions on $K \times [a_{m+1}, b_{m+1}]$, and finish the proof by the induction hypothesis on $\dim T$.

Now we come back to the proof of Proposition 1 in full generality. By Observation 2 and Lemma 1, we can assume that, for each $u \in U$, we have $\dim E_u < m$; in particular, E_u is nowhere dense in K, and the function $K \setminus E_u \ni x \mapsto \dfrac{\partial F}{\partial u_1}(x,u)$ is bounded.

Let us choose a definable function $\lambda = (\lambda_1, \ldots, \lambda_m) : U \to \mathbb{R}^m$ such that for each $u \in U$, $\lambda(u) \in K \setminus E_u$ and

$$\left|\frac{\partial F}{\partial u_1}(\lambda(u), u)\right| \ge \frac{1}{2} \sup_{x \in K \setminus E_u} \left|\frac{\partial F}{\partial u_1}(x,u)\right|. \tag{4}$$

By Observation 2 we can assume that λ is of class $\mathcal{C}^1$ on U. Again due to Observation 2, it suffices to consider the following two cases

Case I:

$$\left|\frac{\partial \lambda_i}{\partial u_1}\right| \leq 1, \quad \text{for each} \quad i \in \{1, ..., m\}; \tag{5}$$

Case II: $\left|\frac{\partial \lambda_1}{\partial u_1}\right| \geq 1$ and $\left|\frac{\partial \lambda_i}{\partial u_1}\right| \leq \left|\frac{\partial \lambda_1}{\partial u_1}\right|$, for each $i \in \{1, ..., m\}$.

In Case I we distinguish the following two subcases (again due to Observations 1-2).

Subcase I.1: $\left|\frac{\partial F}{\partial u_1}(\lambda(u), u)\right| \leq 2mM$.

Then $\left|\frac{\partial F}{\partial u_1}(x, u)\right| \leq 4mM$, for each $x \in K \setminus E_u$ and $u \in U$. Proposition 2 completes the proof in this subcase.

Subcase I.2:

$$\frac{\partial F}{\partial u_1}(\lambda(u), u) \geq 2mM \ \text{ (or, symmetrically, } \leq -2mM) \text{ for each } u \in U. \tag{6}$$

Let us take the new variable $w_1 = F(\lambda(u), u)$ in the place of u_1. Hence $u_1 = \varphi(w_1, \tilde{u})$, where φ is a definable $\mathcal{C}^1$-function and $\tilde{u} = (u_2, ..., u_k)$. Then

$$\frac{\partial w_1}{\partial u_1} = \sum_{i=1}^{m} \frac{\partial F}{\partial x_i} \frac{\partial \lambda_i}{\partial u_1} + \frac{\partial F}{\partial u_1}(\lambda(u), u);$$

whence, by (5), (6) and $\left|\frac{\partial F}{\partial x_i}\right| \leq M$,

$$\frac{\partial w_1}{\partial u_1} \geq \frac{1}{2} \frac{\partial F}{\partial u_1}(\lambda(u), u).$$

Since

$$\frac{\partial}{\partial w_1} F(x, \varphi(w_1, \tilde{u}), \tilde{u}) = \frac{\partial F}{\partial u_1}(x, \varphi(w_1, \tilde{u}), \tilde{u}) \frac{\partial \varphi}{\partial w_1}(w_1, \tilde{u}) = \frac{\frac{\partial F}{\partial u_1}(x, \varphi(w_1, \tilde{u}), \tilde{u})}{\frac{\partial w_1}{\partial u_1}},$$

$$\left|\frac{\partial}{\partial w_1} F(x, \varphi(w_1, \tilde{u}), \tilde{u})\right| \leq \frac{\left|\frac{\partial F}{\partial u_1}(x, \varphi(w_1, \tilde{u}), \tilde{u})\right|}{\left|\frac{\partial w_1}{\partial u_1}\right|} \leq$$

$$2 \frac{\left|\frac{\partial F}{\partial u_1}(x, \varphi(w_1, \tilde{u}), \tilde{u})\right|}{\left|\frac{\partial F}{\partial u_1}(\lambda(\varphi(w_1, \tilde{u}), \tilde{u}), \varphi(w_1, \tilde{u}), \tilde{u})\right|} \leq 4,$$

by (4), for each $x \in K \setminus E_{(w_1, \tilde{u})}$.[5] In order to finish the proof in this subcase, by reducing to Proposition 2, it suffices to use Observations 1-2, Theorem 1 (case $T = \{t_o\}$) and the following lemma.

[5] We denote the subset E in the new variables by the same letter E.

Lemma 3. *Let C be an open definable Lipschitz cell in $\mathbb{R}^k$ with a Lipschitz constant L and let $\varphi = (\varphi_1, ..., \varphi_l) : C \longrightarrow \mathbb{R}^l$ be a $\mathcal{C}^1$-mapping such that $\left|\frac{\partial \varphi_i}{\partial x_j}\right| \leq M$, for each $i \in \{1, ..., l\}$ and $j \in \{1, ..., k\}$.*

Then φ is Lipschitz with a Lipschitz constant depending only on L, M, k and l.

Proof of Lemma 3. By [K; Proposition 8] (or [Paw; Proposition 1]), C is *quasi-convex* with a constant $\tilde{L}$, depending only on L and k ; i.e. for each pair of points $u, w \in C$ there exists a continuous arc $\mu : [0, 1] \longrightarrow A$ such that $\mu(0) = u,\ \mu(1) = y$ and of length $\leq \tilde{L}|u - w|$. Then we finish by [T; p. 76, Remarque 2.5].

In Case II let us introduce the following new variable $v_1 = \lambda_1(u)$ in the place of u_1. Then $u_1 = \psi(v_1, \tilde{u})$, where ψ is a $\mathcal{C}^1$-function. Denote our function F in the new variables by G; i. e. $G(x, v_1, \tilde{u}) := F(x, \psi(v_1, \tilde{u}), \tilde{u})$. Then

$$\frac{\partial G}{\partial v_1}(x, v_1, \tilde{u}) = \frac{\partial F}{\partial u_1}(x, \psi(v_1, \tilde{u}), \tilde{u}) \frac{\partial \psi}{\partial v_1}(v_1, \tilde{u});$$

hence

$$\left|\frac{\partial G}{\partial v_1}(\lambda(\psi(v_1, \tilde{u}), \tilde{u}), v_1, \tilde{u})\right| \geq \frac{1}{2}\left|\frac{\partial G}{\partial v_1}(x, v_1, \tilde{u})\right|,$$

for each $x \in K \setminus E_{(v_1, \tilde{u})}$. Since

$$\lambda(\psi(v_1, \tilde{u}), \tilde{u}) = (v_1, \lambda_2(\psi(v_1, \tilde{u}), \tilde{u}), ..., \lambda_m(\psi(v_1, \tilde{u}), \tilde{u})),$$

and

$$\left|\frac{\partial}{\partial v_1}\lambda_i(\psi(v_1, \tilde{u}), \tilde{u})\right| = \frac{\left|\frac{\partial \lambda_i}{\partial u_1}(\psi(v_1, \tilde{u}), \tilde{u})\right|}{\left|\frac{\partial \lambda_1}{\partial u_1}(\psi(v_1, \tilde{u}), \tilde{u})\right|} \leq 1,$$

for $i \in \{2, ..., m\}$, Case I can be applied which completes the proof in Case II and all the proof of Main Theorem.

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