ON THE COMPLEXITY OF COLLARING THEOREM IN THE LIPSCHITZ CATEGORY

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Abstract

We prove that, if a compact subspace of a metric space can be covered by a family $\mathcal{C}$ of bi–Lipschitz local collars, then it admits also a bi–Lipschitz global collar, whose complexity, measured by the so–called weak bi–Lipschitz constant, depends only on $\mathcal{C}$. If the compactness condition is dropped, a slightly weaker version of this result remains valid, provided $\mathcal{C}$ is uniform in a suitable natural sense. Applications to topological and Lipschitz manifolds are given.

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1 Introduction and main results

In 1962, Morton Brown [1] proved some fundamental results concerning the global triviality of locally trivial embeddings of metric spaces and of topological manifolds.

Let us recall such results. Consider a metric space (X, d) and one of its subspace B.

(i) Then B is called *locally collared* if, for each $b \in B$, there exist an open neighborhood U of b in B, an open neighborhood Ω of b in X and a homeomorphism $h : U \times [0, 1) \longrightarrow \Omega$ such that $U = B \cap \Omega$ and $h(b, 0) = b$ for each $b \in U$. If there exists an open neighborhood Ω' of B in X and a homeomorphism $H : B \times [0, 1) \longrightarrow \Omega'$ such that $H(b, 0) = b$ for each $b \in B$, then B is said to be *collared* and H is called *collar of* B.

(ii) Similarly, B is called *locally bi–collared* if, for each $b \in B$, there exist an open neighborhood U of b in B, an open neighborhood Ω of b in X and a homeomorphism $g : U \times (-1, 1) \longrightarrow \Omega$ such that $U = B \cap \Omega$ and $g(b, 0) = b$ for each $b \in U$. If there exist an open neighborhood Ω' of B in X and a homeomorphism $G : B \times (-1, 1) \longrightarrow \Omega'$ such that $G(b, 0) = b$ for each $b \in B$, then B is said to be *bi–collared* and G is called *bi–collar of* B.

(iii) Suppose now that X is second countable and it is equipped with a structure of topological manifold (without boundary) of positive dimension m and B is a topological hypersurface of X; that is, a non–empty closed subset of X, which becomes a topological manifold of dimension $m - 1$ when it is equipped with the relative topology. We call B *two–sided* if it is connected and there exists a connected open neighborhood W of B in X such that $W \setminus B$ has two connected components. For each positive integer m, we identify $\mathbb{R}^{m-1}$ with $\mathbb{R}^{m-1} \times \{0\}$ in $\mathbb{R}^{m-1} \times \mathbb{R} = \mathbb{R}^m$. We say that B is *locally flat* if, for each $b \in B$, there exist an open neghborhood Ω of b in X, an open subset O of $\mathbb{R}^m$ and a homeomorphism $\varphi : \Omega \longrightarrow O$ such that $\varphi(B \cap \Omega) = \mathbb{R}^{m-1} \cap O$. This is equivalent to say that B is locally bi–collared.

We are in position to recall the statements of the first three theorems of [1]. We shall follow the original numeration.

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Theorem 1 (The Collaring Theorem). *Every locally collared subspace of a metric space is collared.*

Theorem 2 *The boundary of a topological manifold with boundary has a collar.*

Theorem 3 *Every locally flat two–sided topological hypersurface of a topological manifold has a bi–collar.*

Let (Y, e) and (Z, q) be metric spaces and let $f : Y \longrightarrow Z$ be a map between them. We recall that f is said to be *Lipschitz* if there exists a real constant c in $\mathbb{R}^+ := \{t \in \mathbb{R} \,|\, t \geq 0\}$ such that

$$q(f(x), f(y)) \leq c \cdot e(x, y) \quad \text{for each} \quad x, y \in Y. \tag{1}$$

The map f is called *locally Lipschitz* (LIP for short) if, for each $y \in Y$, there exists a neighborhood U of y in Y such that the restriction $f|_U$ of f to U is Lipschitz.

Suppose now that $f : Y \longrightarrow Z$ is a homeomorphism. Then f is called *bi–Lipschitz* if both f and f^{-1} are Lipschitz maps. If both f and f^{-1} are LIP maps only, then f is said to be a *lipemorphism.*

Consider the metric space (X, d) and its subspace B again. Define the metric D on $X \times \mathbb{R}$ by setting

$$D\big((x, t), (y, s)\big) := \big(d(x, y)^2 + (t - s)^2\big)^{1/2} \tag{2}$$

for each $(x, t), (y, s) \in X \times \mathbb{R}$. Equip each subspace of $X \times \mathbb{R}$ with the metric induced by D.

A lipeomorphism $H : B \times [0, 1) \longrightarrow \Omega'$ from $B \times [0, 1)$ to an open neighborhood Ω' of B in X is called LIP *collar of* B if $H(b, 0) = b$ for each $b \in B$. If there exists a lipeomorphism $G : B \times (-1, 1) \longrightarrow \Omega'$ with $G(b, 0) = b$ for each $b \in B$, then we call G LIP *bi–collar of* B. If, for each $b \in B$, there exist an open neighborhood U of b in B, which has a LIP collar, then we say that B is *bi–Lipschitz locally collared.* Observe that B is bi–Lipschitz locally collared if and only if it is locally collared in the sense of above point (i), where the homeomorphisms h are required to be bi–Lipschitz.

Suppose that X is a metric topological manifold; that is, a second countable metric space equipped with a structure of topological manifold. Let B be a topological hypersurface of X. According to above point (iii), we say that B is *bi–Lipschitz locally flat* if, for each $b \in B$, there exist an open neghborhood Ω of b in X, an open subset O of $\mathbb{R}^m$ and a lipeomorphism (or, equivalently, a bi–Lipschitz homeomorphism) $\varphi : \Omega \longrightarrow O$ such that $\varphi(B \cap \Omega) = \mathbb{R}^{m-1} \cap O$. Observe that B is bi–Lipschitz locally flat if and only if it is locally bi–collared in the sense of above point (ii), where the homeomorphisms g are required to be bi–Lipschitz (see the proof of Theorems 3L and 3L* below).

We fix a convention: the empty subset of X is neither bi–Lipschitz locally collared nor bi–Lipschitz locally flat.

Let us specify the meaning we give to the notion of "metric Lipschitz manifold with boundary". A *metric Lipschitz manifold with boundary* of dimension m is a second countable metric space in which each point has a closed neighborhood bi–Lipschitz homeomorphic to the standard unit closed ball of $\mathbb{R}^m$. As is usual, the boundary is formed by the points which do not have open neighborhoods bi–Lipschitz homeomorphic to the standard unit open ball of $\mathbb{R}^m$. We always assume that the boundary of a metric Lipschitz manifold with boundary is non–empty. If the boundary of a metric Lipschitz manifold with boundary is compact, then we call such a manifold *metric Lipschitz manifold with compact boundary.*

In Section 7 of [5], J. Luukkainen and J. Väisälä proved the following LIP versions of above Theorems 1 and 2.

Theorem 1$'$ *Every bi–Lipschitz locally collared subspace of a metric space has* LIP *collars.*

Theorem 2$'$ *The boundary of a metric Lipschitz manifold with boundary has* LIP *collars.*

Combining the latter result with the argument we shall use in Section 2 to prove our Theorems 3L and 3L*, one obtains:

Theorem 3′ *Every bi–Lipschitz locally flat two–sided topological hypersurface of a metric topological manifold has* LIP *bi–collars.*

The aim of this paper is to improve the preceding three theorems by giving explicit estimates of the complexity of LIP collars and LIP bi–collars mentioned in the statements. We shall measure such a complexity by means of the notion of weak bi–Lipschitz constant.

In order to present our results, we need some preparations.

Equip each vector space $\mathbb{R}^m$ with the usual euclidean metric. Let $\mathbb{R}^+_* := \mathbb{R}^+ \setminus \{0\}$, let $\mathbb{R}^- := \mathbb{R} \setminus \mathbb{R}^+_*$ and let $\mathbb{R}^-_* := \mathbb{R}^- \setminus \{0\}$. For each point y of the metric space (Y, e) and for each $r \in \mathbb{R}^+$, we denote by $\mathcal{B}_e(y, r)$ the open ball of (Y, e) centered at y with radius r; that is, $\mathcal{B}_e(y, r) := \{x \in Y \,|\, e(x, y) < r\}$. Evidently, if $r = 0$, then $\mathcal{B}_e(y, r)$ is empty. Let $f : Y \longrightarrow Z$ be a map between the metric spaces (Y, e) and (Z, q). If f is Lipschitz, then the infimum of the real constants c in $\mathbb{R}^+$ with property (1), denoted by $\mathrm{Lip}(f)$, is called *Lipschitz constant of* f. Suppose f is LIP. Let $y \in Y$ and let r be a positive real number (depending on y and on f) such that $f|_{\mathcal{B}_e(y,r)}$ is Lipschitz. We define the *pointwise Lipschitz constant* $\mathrm{Lip}_y(f)$ *of* f *at* y by setting

$$\mathrm{Lip}_y(f) := \inf_{s\in(0,r]} \mathrm{Lip}\left(f|_{\mathcal{B}_e(y,s)}\right)$$

and the *weak Lipschitz constant* $\mathrm{Lip}^*(f)$ *of* f by setting

$$\mathrm{Lip}^*(f) := \sup_{y\in Y} \mathrm{Lip}_y(f).$$

We remark that the definition of $\mathrm{Lip}_y(f)$ is consistent, because it does not depend on the choice of r. We say that f is *weakly Lipschitz* if it is LIP and its weak Lipschitz constant is finite. Evidently, if f is Lipschitz, then it is also weakly Lipschitz and $\mathrm{Lip}^*(f) \leq \mathrm{Lip}(f)$.

A simple remark on the latter notions is in order. If Y is compact, then every LIP map from (Y, e) to (Z, q) is Lipschitz (see point (2) of Lemma 2.3) and hence it is weakly Lipschitz. If the domain is not compact, then the situation may be completely different. For example, the function from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$, which sends x into $x/|x|$, is weakly Lipschitz, but not Lipschitz. Furthermore, the function from $\mathbb{R}$ into itself, which sends x into x^2 is LIP, but not weakly Lipschitz. Suppose now that Y is a non–empty open subset of some $\mathbb{R}^m$, Z coincides with $\mathbb{R}$ and $f : Y \longrightarrow \mathbb{R}$ is a Lipschitz function. Then, by Rademacher's Theorem, f has weak gradient $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_m)$ in $L^\infty(Y, \mathbb{R}^m)$ and it is immediate to verify that $\|\nabla f\|_{L^\infty(Y,\mathbb{R}^m)} := \|(\sum_{i=1}^m (\partial f/\partial x_i)^2)^{1/2}\|_{L^\infty(Y,\mathbb{R})}$ is less than or equal to $\mathrm{Lip}^*(f)$. Finally, if Y is convex, then $\mathrm{Lip}^*(f)$ and $\mathrm{Lip}(f)$ coincide.

Suppose that $f : Y \longrightarrow Z$ is a homeomorphism from Y to Z. If f is bi–Lipschitz, then we define the *bi–Lipschitz constant* $\mathrm{biLip}(f)$ *of* f by setting

$$\mathrm{biLip}(f) := \max\{1, \mathrm{Lip}(f)\} \cdot \max\{1, \mathrm{Lip}(f^{-1})\}.$$

We call f *weakly bi–Lipschitz* if both f and f^{-1} are weakly Lipschitz. In this situation, we define the *weak bi–Lipschitz constant* $\mathrm{biLip}^*(f)$ *of* f by setting

$$\mathrm{biLip}^*(f) := \max\{1, \mathrm{Lip}^*(f)\} \cdot \max\{1, \mathrm{Lip}^*(f^{-1})\}.$$

Evidently, if f is a weakly bi–Lipschitz homeomorphism, then $\mathrm{biLip}^*(f) = \mathrm{biLip}^*(f^{-1})$. In Corollary 2.2 below, we shall see that, if Y contains at least two points, then $\mathrm{Lip}^*(f) \cdot \mathrm{Lip}^*(f^{-1}) \geq 1$ and hence $\mathrm{biLip}^*(f) = \max\{\mathrm{Lip}^*(f), \mathrm{Lip}^*(f^{-1}), \mathrm{Lip}^*(f) \cdot \mathrm{Lip}^*(f^{-1})\}$. If f is a bi–Lipschitz homeomorphism, then $\mathrm{biLip}(f) = \mathrm{biLip}(f^{-1})$ and $\mathrm{biLip}^*(f) \leq \mathrm{biLip}(f)$. Moreover, $\mathrm{biLip}(f) = \max\{\mathrm{Lip}(f), \mathrm{Lip}(f^{-1}), \mathrm{Lip}(f) \cdot \mathrm{Lip}(f^{-1})\}$, provided Y contains at least two points.

Consider the metric space (X,d) and its subspace B once again.

We introduce some additional Lipschitz version of the notions of collar and bi–collar.

We say that the subspace B of X is *bi–Lipschitz collared* if it is collared in the sense of above point (i), where the homeomorphism H is required to be bi–Lipschitz. We call such a bi–Lipschitz homeomorphism H *bi–Lipschitz collar of* B. If we require that the homeomorphism H is weakly bi–Lipschitz only, then we say that H is a *weakly bi–Lipschitz collar of* B.

Similarly, we say that B is *bi–Lipschitz bi–collared* if it is bi–collared in the sense of above point (ii), where the homeomorphism G is required to be bi–Lipschitz. We call such a bi–Lipschitz homeomorphism G *bi–Lipschitz bi–collar of* B. If we require that the homeomorphism G is weakly bi–Lipschitz only, then we say that G is a *weakly bi–Lipschitz bi–collar of* B.

We need also some notions regarding finite open coverings of a metric space.

Given any non–empty subset A of X, we denote by $d_A : X \longrightarrow \mathbb{R}^+$ the function, which sends $x \in X$ into $d(x,A) := \inf_{y \in A} d(x,y) \in \mathbb{R}^+$. A simple exercise ensures that d_A is a Lipschitz function on X with Lipschitz constant ≤ 1, $d_A^{-1}(0)$ is equal to the closure $\overline{A}$ of A in X and $d_A = d_{\overline{A}}$. Let $\mathcal{P} = \{P_n\}_{n=1}^k$ be a finite open covering of X (we implicitly assume $k \geq 1$). We say that $\mathcal{P}$ is *proper* if $\emptyset \neq P_n \neq X$ for each $n \in \{1,\ldots,k\}$. This notion of properness will be used below to avoid trivial situations. Suppose $\mathcal{P}$ is proper. The *Lebesgue function of* $\mathcal{P}$ is the function $\mathcal{L}(\mathcal{P}) : X \longrightarrow \mathbb{R}^+$ defined as follows:

$$\mathcal{L}(\mathcal{P}) := \max\{d_{X \setminus P_1}, \ldots, d_{X \setminus P_k}\}.$$

It is easy to show, by induction on k, that $\mathcal{L}(\mathcal{P})$ is Lipschitz with Lipschitz constant ≤ 1. We say that $\mathcal{P}$ is a *Lebesgue uniform covering of* X if the infimum $\ell(P)$ of $\mathcal{L}(P)$ is positive. This is equivalent to the existence of a Lebesgue number of $\mathcal{P}$; that is, a positive real number δ with the following property: for each $x \in X$, the open ball of X centered at x with radius δ is contained in at least one element P_n of $\mathcal{P}$. If $\mathcal{P}$ is Lebesgue uniform, then $\ell(P)$ is the greatest Lebesgue number of $\mathcal{P}$. We remark that, by definition, a Lebesgue uniform covering of X is assumed to be finite, open and proper. Evidently, if X is compact, every finite proper open covering of X is Lebesgue uniform. We call *internal radius of* $\mathcal{P}$ the supremum $r(\mathcal{P}) \in \mathbb{R}^+ \cup \{+\infty\}$ of $\mathcal{L}(\mathcal{P})$. If $r(\mathcal{P})$ is finite, then it is the greatest positive real number R with the following property: there exist $n \in \{1,\ldots,k\}$ and $x \in X$ such that the open ball of X centered at x with radius R is contained in P_n.

Finally, we give some definitions concerning the existence of local collars and bi–collars.

Suppose that the subspace B of X is bi–Lipschitz locally collared. Let $\mathcal{U} = \{U_n\}_{n=1}^k$ be a finite proper open covering of B, let $\{\Omega_n\}_{n=1}^k$ be a family of open subsets of X such that $U_n = B \cap \Omega_n$ for each $n \in \{1,\ldots,k\}$ and let $\mathbf{h} := \{h_n : U_n \times [0,1) \longrightarrow \Omega_n\}_{n=1}^k$ be a family of homeomorphisms. Then we call the pair $(\mathcal{U},\mathbf{h})$ *system of bi–Lipschitz local collars of* B if each h_n is a bi–Lipschitz homeomorphism such that $h_n(b,0) = b$ for each $b \in U_n$. Such a system $(\mathcal{U},\mathbf{h})$ is said to be *Lebesgue uniform* if $\mathcal{U}$ is.

Suppose now that X is a metric topological manifold, B is a bi–Lipschitz locally flat topological hypersurface of X and $\mathbf{g} = \{g_n : U_n \times (-1,1) \longrightarrow \Omega_n\}_{n=1}^k$ is a family of bi–Lipschitz homeomorphisms such that $g_n(b,0) = b$ for each $n \in \{1,\ldots,k\}$ and for each $b \in U_n$. In this situation, we say that the pair $(\mathcal{U},\mathbf{g})$ is a *system of bi–Lipschitz local bi–collars of* B. Such a system $(\mathcal{U},\mathbf{g})$ is said to be *Lebesgue uniform* if $\mathcal{U}$ is. Let B be two–sided. Then we say that the system $(\mathcal{U},\mathbf{g})$ is *two–sided coherent* if there exists a connected open neighborhood Z of B such that $Z \setminus B$ has two connected components and $\bigcup_{n=1}^k \Omega_n \subset Z$. Suppose, in addition, that B is compact. If the system $(\mathcal{U},\mathbf{g})$ is not two–sided coherent, then it is easy to construct one system, which is. In fact, B is two–sided so there exists a connected open neighborhood W of B such that $W \setminus B$ has two connected components. Choose a small positive real number ε and, for each $n \in \{1,\ldots,k\}$, define the open subset Ω_n^* of X by $\Omega_n^* := g_n(U_n \times (-\varepsilon,\varepsilon))$ and the bi–Lipschitz homeomorphism $g_n^* : U_n \times (-1,1) \longrightarrow \Omega_n^*$ by $g_n^*(x,t) := g_n(x,\varepsilon t)$. Thanks to the compactness of B, if ε is sufficiently small, then $(\{U_n\}_{n=1}^k, \{g_n^*\}_{n=1}^k)$ is a two–sided coherent system of bi–Lipschitz local bi–collars of B.

We are now in position to present our Lipschitz versions of the above three theorems of Brown.

Theorem 1L *If a subspace B of a metric space has a Lebesgue uniform system $(\mathcal{U}, \mathbf{h})$ of bi–Lipschitz local collars, then it has also a weakly bi–Lipschitz collar.*

More precisely, the following assertion holds. Let $(\mathcal{U}, \mathbf{h}) = (\{U_n\}_{n=1}^k, \{h_n\}_{n=1}^k)$. Define $L := \min\{1, \ell(\mathcal{U})\}$ and $R := \min\{1, r(\mathcal{U})\}$. Then B has a weakly bi–Lipschitz collar $H : B \times [0, 1) \longrightarrow \Omega$, having upper bounds for $\mathrm{Lip}^*(H)$, $\mathrm{Lip}^*(H^{-1})$ and $\mathrm{biLip}^*(H)$ in terms of $(\mathcal{U}, \mathbf{h})$ only, as the following:

$$\mathrm{Lip}^*(H) \leq (\sqrt{2})^{3k+3} \cdot (3 + 2R)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(h_n), \tag{3}$$

$$\mathrm{Lip}^*(H^{-1}) \leq (\sqrt{2})^{3k+1} \cdot (1 + L^{-1}) \cdot (3 + 2R)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(h_n) \tag{4}$$

and hence

$$\mathrm{biLip}^*(H) \leq 2^{3k+2} \cdot (1 + L^{-1}) \cdot (3 + 2R)^{2k} \cdot (\textstyle\prod_{n=1}^k \mathrm{biLip}^*(h_n))^2. \tag{5}$$

Moreover, H extends to a weakly bi–Lipschitz homeomorphism from $B \times [0, 1]$ to the closure of Ω in X.

Theorem 1L* (compact case) *Every bi–Lipschitz locally collared compact subspace of a metric space is bi–Lipschitz collared.*

More precisely, the following assertion holds. Let X be a metric space, let B be a bi–Lipschitz locally collared compact subspace of X and let $(\mathcal{U}, \mathbf{h}) = (\{U_n\}_{n=1}^k, \{h_n\}_{n=1}^k)$ be a system of bi–Lipschitz local collars of B. Define $L := \min\{1, \ell(\mathcal{U})\}$ and $R := \min\{1, r(\mathcal{U})\}$. Then B has a bi–Lipschitz collar $H : B \times [0, 1) \longrightarrow \Omega$, which satisfies inequalities (3), (4) and (5), and extends to a bi–Lipschitz homeomorphism from $B \times [0, 1]$ to the closure of Ω in X.

Theorem 2L *Let X be a metric Lipschitz manifold with boundary of positive dimension m and let B be its boundary. Suppose that there exist a Lebesgue uniform covering $\mathcal{U} = \{U_n\}_{n=1}^k$ of B, a family $\{\Omega_n\}_{n=1}^k$ of open subsets of X such that $U_n = B \cap \Omega_n$ for each $n \in \{1, \ldots, k\}$, a family $\{V_n\}_{n=1}^k$ of open subsets of $\mathbb{R}^{m-1}$ and, for each $n \in \{1, \ldots, k\}$, a Lipschitz chart $\phi_n : \Omega_n \longrightarrow V_n \times [0, 1)$ of X relative to B; that is, ϕ_n is a bi–Lipschitz homeomorphism such that $\phi_n(U_n) = V_n \times \{0\}$. Then B has a weakly bi–Lipschitz collar.*

More precisely, the following assertion holds. Define $L := \min\{1, \ell(\mathcal{U})\}$ and $R := \min\{1, r(\mathcal{U})\}$. Then B has a weakly bi–Lipschitz collar $H : B \times [0, 1) \longrightarrow \Omega$, having upper bounds for $\mathrm{Lip}^*(H)$, $\mathrm{Lip}^*(H^{-1})$ and $\mathrm{biLip}^*(H)$ in terms of $\mathcal{U}$ and of the charts $\{\phi_n\}_{n=1}^k$ only, as the following:

$$\mathrm{Lip}^*(H) \leq (\sqrt{2})^{5k+3} \cdot (3 + 2R)^k \cdot \left(\textstyle\prod_{n=1}^k \mathrm{biLip}^*(\phi_n)\right)^2, \tag{6}$$

$$\mathrm{Lip}^*(H^{-1}) \leq (\sqrt{2})^{5k+1} \cdot (1 + L^{-1}) \cdot (3 + 2R)^k \cdot \left(\textstyle\prod_{n=1}^k \mathrm{biLip}^*(\phi_n)\right)^2 \tag{7}$$

and hence

$$\mathrm{biLip}^*(H) \leq 2^{5k+2} \cdot (1 + L^{-1}) \cdot (3 + 2R)^{2k} \cdot \left(\textstyle\prod_{n=1}^k \mathrm{biLip}^*(\phi_n)\right)^4. \tag{8}$$

Moreover, H extends to a weakly bi–Lipschitz homeomorphism from $B \times [0, 1]$ to the closure of Ω in X.

Theorem 2L* (compact case) *The boundary of a metric Lipschitz manifold with compact boundary has a bi–Lipschitz collar.*

More precisely, the following assertion holds. Let X be a metric Lipschitz manifold with compact boundary of positive dimension m, let $\mathcal{U} = \{U_n\}_{n=1}^k$ be a finite proper open covering of the boundary B of X, let $\{\Omega_n\}_{n=1}^k$ be a family of open subsets of X such that $U_n = B \cap \Omega_n$ for each $n \in \{1, \ldots, k\}$, let $\{V_n\}_{n=1}^k$ be a family of open subsets of $\mathbb{R}^{m-1}$ and, for each $n \in \{1, \ldots, k\}$, let $\phi_n : \Omega_n \longrightarrow V_n \times [0, 1)$ be a Lipschitz chart of X relative to B. Define $L := \min\{1, \ell(\mathcal{U})\}$ and $R := \min\{1, r(\mathcal{U})\}$. Then B has a bi–Lipschitz collar $H : B \times [0, 1) \longrightarrow \Omega$, which satisfies inequalities (6), (7) *and* (8), *and extends to a bi–Lipschitz homeomorphism from $B \times [0, 1]$ to the closure of Ω in X.*

Theorem 3L *If a two–sided topological hypersurface B of a metric topological manifold has a two–sided coherent Lebesgue uniform system of bi–Lipschitz local bi–collars, then it has also a weakly bi–Lipschitz bi–collar.*

More precisely, the following assertion holds. Let X be a metric topological manifold of positive dimension, let B be a bi–Lipschitz locally flat two–sided topological hypersurface of X and let $(\mathcal{U}, \mathbf{g}) = (\{U_n\}_{n=1}^k, \{g_n\}_{n=1}^k)$ be a two–sided coherent Lebesgue uniform system of bi–Lipschitz local bi–collars of B. Define $L := \min\{1, \ell(\mathcal{U})\}$, $R := \min\{1, r(\mathcal{U})\}$ and $S := \max_{n\in\{1,\ldots,k\}} \mathrm{biLip}^*(g_n)$. Then B has a weakly bi–Lipschitz bi–collar $G : B \times (-1, 1) \longrightarrow \Omega$, having upper bounds for $\mathrm{Lip}^*(G)$, $\mathrm{Lip}^*(G^{-1})$ *and* $\mathrm{biLip}^*(G)$ *in terms of $(\mathcal{U}, \mathbf{g})$ only, as the following:*

$$\mathrm{Lip}^*(G) \le (\sqrt{2})^{3k+4} \cdot (3 + 2R)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(g_n), \tag{9}$$

$$\mathrm{Lip}^*(G^{-1}) \le (\sqrt{2})^{3k+2} \cdot (1 + L^{-1}) \cdot (3 + 2R)^k \cdot S \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(g_n) \tag{10}$$

and hence

$$\mathrm{biLip}^*(G) \le 2^{3k+3} \cdot (1 + L^{-1}) \cdot (3 + 2R)^{2k} \cdot S \cdot \left(\textstyle\prod_{n=1}^k \mathrm{biLip}^*(g_n)\right)^2. \tag{11}$$

Moreover, G extends to a weakly bi–Lipschitz homeomorphism from $B \times [-1, 1]$ to the closure of Ω in X.

Theorem 3L* (compact case) *Every bi–Lipschitz locally flat two–sided compact topological hypersurface of a metric topological manifold is bi–Lipschitz bi–collared.*

More precisely, the following assertion holds. Let X be a metric topological manifold of positive dimension, let B be a bi–Lipschitz locally flat two–sided compact topological hypersurface of X and let $(\mathcal{U}, \mathbf{g}) = (\{U_n\}_{n=1}^k, \{g_n\}_{n=1}^k)$ be a two–sided coherent system of bi–Lipschitz local bi–collars of B. Define $L := \min\{1, \ell(\mathcal{U})\}$, $R := \min\{1, r(\mathcal{U})\}$ and $S := \max_{n\in\{1,\ldots,k\}} \mathrm{biLip}^*(g_n)$. Then B has a bi–Lipschitz bi–collar $G : B \times (-1, 1) \longrightarrow \Omega$, which satisfies inequalities (9), (10) *and* (11), *and extends to a bi–Lipschitz homeomorphism from $B \times [-1, 1]$ to the closure of Ω in X.*

Let us present a significant consequence of Theorem 3L*.

Let m be a positive integer and let Y be a topological hypersurface of $\mathbb{R}^m$. We say that Y is an *embedded Lipschitz hypersurface of* $\mathbb{R}^m$ if it is bi–Lipschitz locally flat or, equivalently, if, for each $y \in Y$, there exist an open neighborhood Ω of y in $\mathbb{R}^m$, an open subset V of $\mathbb{R}^{m-1}$ and a bi–Lipschitz homeomorphism $\varphi : \Omega \longrightarrow V \times (-1, 1)$ such that $\varphi(Y \cap \Omega) = V \times \{0\}$.

Theorem 3L** *For each positive integer m, every compact connected embedded Lipschitz hypersurface of $\mathbb{R}^m$ is bi–Lipschitz bi–collared.*

More precisely, the following assertion holds. Let Y be a compact connected embedded Lipschitz hypersurface of $\mathbb{R}^m$, where m is a positive integer. Let $\mathcal{U} = \{U_n\}_{n=1}^k$ be a finite proper open covering of Y, let $\{\Omega_n\}_{n=1}^k$ be a family of open subsets of $\mathbb{R}^m$ such that $U_n = Y \cap \Omega_n$ for each $n \in \{1, \ldots, k\}$, let $\{V_n\}_{n=1}^k$ be a family of open subsets of $\mathbb{R}^{m-1}$ and, for each $n \in \{1, \ldots, k\}$, let $\varphi_n : \Omega_n \longrightarrow V_n \times (-1, 1)$ be a bi–Lipschitz homeomorphism *such that* $\varphi_n(U_n) = V_n \times \{0\}$. *Define* $L := \min\{1, \ell(\mathcal{U})\}$, $R := \min\{1, r(\mathcal{U})\}$ *and* $U := \max_{n\in\{1,\ldots,k\}} \mathrm{biLip}^*(\varphi_n)$. *Then* Y *has a bi–Lipschitz collar* $G : Y \times (-1, 1) \longrightarrow \Omega$, *having upper bounds for* $\mathrm{Lip}^*(G)$, $\mathrm{Lip}^*(G^{-1})$ *and* $\mathrm{biLip}^*(G)$ *in terms of* $\mathcal{U}$ *and the* $\{\varphi_n\}_{n=1}^k$*'s only, as the following:*

$$\mathrm{Lip}^*(G) \le (\sqrt{2})^{5k+4} \cdot (3 + 2R)^k \cdot \left(\prod_{n=1}^k \mathrm{biLip}^*(\varphi_n)\right)^2,$$
$$\mathrm{Lip}^*(G^{-1}) \le (\sqrt{2})^{5k+4} \cdot (1 + L^{-1}) \cdot (3 + 2R)^k \cdot U^2 \cdot \left(\prod_{n=1}^k \mathrm{biLip}^*(\varphi_n)\right)^2$$

and hence

$$\mathrm{biLip}^*(G) \le 2^{5k+4} \cdot (1 + L^{-1}) \cdot (3 + 2R)^{2k} \cdot U^2 \cdot \left(\prod_{n=1}^k \mathrm{biLip}^*(\varphi_n)\right)^4.$$

Moreover, G *extends to a bi–Lipschitz homeomorphism from* $B \times [-1, 1]$ *to the closure of* Ω *in* $\mathbb{R}^m$*.*

It is well–known that every LIP submanifold B of $\mathbb{R}^m$ has a LIP tubular neighborhood in the following sense: there exist an open neighborhood T of B in $\mathbb{R}^m$ and a LIP map $\Pi : T \longrightarrow B$ such that $\Pi(b) = b$ for each $b \in B$ (see Theorem 5.13 of [5]).

Let us introduce the analogous notions of Lipschitz and weakly Lipschitz tubular neighborhoods in the hypersurface case.

Let X be a metric topological manifold and let B be a topological hypersurface of X. Given an open neighborhood Ω of B in X and a map $\rho : \Omega \longrightarrow B$, we say that ρ is a *Lipschitz tubular neighborhood of* B (resp. a *weakly Lipschitz tubular neighborhood of* B) if ρ is Lipschitz (resp. weakly Lipschitz) and $\rho(b) = b$ for each $b \in B$.

As easy consequences of Theorems 3L and 3L*, we obtain:

Theorem 4L *If a two–sided topological hypersurface* B *of a metric topological manifold has a two–sided coherent Lebesgue uniform system of bi–Lipschitz local bi–collars, then it has also a weakly Lipschitz tubular neighborhood.*

Moreover, if G *is a weakly bi–Lipschitz bi–collar of* B (*which exists by Theorem* 3L), *then there exists a weakly Lipschitz tubular neighborhood* ρ *of* B *such that*

$$\mathrm{Lip}^*(\rho) \le \mathrm{Lip}^*(G^{-1}).$$

Theorem 4L* (compact case) *Every bi–Lipschitz locally flat two–sided compact topological hypersurface* B *of a metric topological manifold* (*as, for example, a compact connected embedded Lipschitz hypersurface of some* $\mathbb{R}^m$) *has Lipschitz tubular neighborhoods.*

Moreover, if G *is a bi–Lipschitz bi–collar of* B (*which exists by Theorem* 3L*), *then there exists a Lipschitz tubular neighborhood* ρ *of* B *such that*

$$\mathrm{Lip}^*(\rho) \le \mathrm{Lip}^*(G^{-1}).$$

The proofs of the preceding theorems are given in the next section.

For some applications of Lipschitz collaring results contained in Section 7 of [5] and in this paper, we refer the reader to the final version of [3], which will appear soon.

We wish to thank Silvano Delladio and Francesco Serra Cassano for useful discussions.

2 Proofs

Some preliminary lemmas. Let (X, d) be a metric space. Let S and T be two non–empty subsets of X. We recall that the diameter $\mathrm{Diam}_d(S)$ of S in X and the distance $d(S, T)$ between S and T in X are defined as follows:

$$\mathrm{Diam}_d(S) := \sup_{x,y\in S} d(x, y) \quad \text{and} \quad d(S, T) := \inf_{x\in S,\, y\in T} d(x, y).$$

If S is compact, then $\mathrm{Diam}_d(S)$ is finite. Furthermore, if, in addition, T is closed and disjoint from S, then $d(S,T)$ is positive.

We collect below some easy, but useful, results.

Lemma 2.1 *Let $f:(X,d)\longrightarrow(Y,e)$ and $g:(Y,e)\longrightarrow(Z,q)$ be maps between metric spaces. The following assertions hold.*

(1) *If f and g are Lipschitz, then $g\circ f$ is also Lipschitz and $\mathrm{Lip}(g\circ f)\le\mathrm{Lip}(g)\cdot\mathrm{Lip}(f)$.*

(2) *If f and g are LIP, then $g\circ f$ is also LIP and $\mathrm{Lip}_x(g\circ f)\le\mathrm{Lip}_{f(x)}(g)\cdot\mathrm{Lip}_x(f)$ for each $x\in X$. In particular, if f and g are weakly Lipschitz, then $g\circ f$ is also weakly Lipschitz and $\mathrm{Lip}^*(g\circ f)\le\mathrm{Lip}^*(g)\cdot\mathrm{Lip}^*(f)$.*

Proof. The proof is elementary. First, suppose that f and g are Lipschitz. For each $x,y\in X$, we have:

$$q\big((g\circ f)(x),(g\circ f)(y)\big)\le\mathrm{Lip}(g)e(f(x),f(y))\le\mathrm{Lip}(g)\mathrm{Lip}(f)d(x,y).$$

(1) is proved. Suppose now that f and g are LIP. Let $x\in X$. Choose $r\in\mathbb{R}^+_*$ in such a way that $g|_{\mathcal{B}_e(f(x),r)}$ is Lipschitz. Since f is continuous at x, there exists $s\in\mathbb{R}^+_*$ such that $f(\mathcal{B}_d(x,s))\subset\mathcal{B}_e(f(x),r)$. It follows that $g\circ f$ is LIP. Moreover, proceeding as above, we obtain: $\mathrm{Lip}((g\circ f)|_{\mathcal{B}_d(x,s)})\le\mathrm{Lip}(g|_{\mathcal{B}_e(f(x),r)})\cdot\mathrm{Lip}(f|_{\mathcal{B}_d(x,s)})$ and hence

$$\mathrm{Lip}_x(g\circ f)\le\inf_{r\in\mathbb{R}^+_*}\mathrm{Lip}(g|_{\mathcal{B}_e(f(x),r)})\cdot\mathrm{Lip}_x(f)=\mathrm{Lip}_{f(x)}(g)\cdot\mathrm{Lip}_x(f)$$

The latter inequality implies the last part of point (2). □

Corollary 2.2 *Let $f:(X,d)\longrightarrow(Y,e)$ be a lipeomorphism between metric spaces. If X contains at least two points, then $\mathrm{Lip}^*(f)\cdot\mathrm{Lip}^*(f^{-1})\ge1$.*

Proof. Denote by id_X the identity map on X. Since X has at least two points, it follows that $\mathrm{Lip}^*(\mathrm{id}_X)=1$. By point (2) of Lemma 2.1, we infer that

$$1=\mathrm{Lip}^*(\mathrm{id}_X)=\mathrm{Lip}^*(f^{-1}\circ f)\le\mathrm{Lip}^*(f^{-1})\cdot\mathrm{Lip}^*(f).\ \square$$

Lemma 2.3 *Let $f:(X,d)\longrightarrow(Y,e)$ be a map between metric spaces such that $\mathrm{Diam}_e(f(X))$ is finite. The following assertions hold.*

(1) *Let U be a non–empty subset of X and let V be a proper subset of X contained in U such that $d(V,X\setminus U)$ is positive and the restrictions $f|_U$ of f to U and $f|_{X\setminus V}$ of f to $X\setminus V$ are Lipschitz. Then f is Lipschitz. Moreover, if η denotes $d(V,X\setminus U)$, then we have that*

$$\mathrm{Lip}(f)\le\max\left\{\mathrm{Lip}(f|_U),\mathrm{Lip}(f|_{X\setminus V}),\frac{\mathrm{Diam}_e(f(X))}{\eta}\right\}.$$

(2) *Let $\mathcal{P}=\{P_n\}_{n=1}^k$ be a Lebesgue uniform covering of X such that the restriction $f|_{P_n}$ of f to each P_n is Lipschitz and let $M:=\max_{n\in\{1,\dots,k\}}\mathrm{Lip}(f|_{P_n})$. Then f is Lipschitz and we have that*

$$\mathrm{Lip}(f)\le\max\left\{M,\frac{\mathrm{Diam}_e(f(X))}{\ell(\mathcal{P})}\right\}.$$

Proof. (1) Let $x,y\in X$. If $d(x,y)\ge\eta$, then

$$e(f(x),f(y))\le\frac{e(f(x),f(y))}{d(x,y)}\cdot d(x,y)\le\frac{\mathrm{Diam}_e(f(X))}{\eta}\cdot d(x,y).$$

If $d(x,y) < \eta$, then both points x and y are contained either in U or in $X \setminus V$. In this situation, we infer that $e(f(x), f(y)) \leq \max\{\mathrm{Lip}(f|_U), \mathrm{Lip}(f|_{X\setminus V})\} \cdot d(x,y)$. It follows that $\mathrm{Lip}(f) \leq \max\{\mathrm{Lip}(f|_U), \mathrm{Lip}(f|_{X\setminus V}), \mathrm{Diam}_e(f(X))/\eta\}$.

(2) Let us proceed as above. Choose $x, y \in X$. If $d(x,y) \geq \ell(\mathcal{P})$, then we have that $e(f(x), f(y)) \leq \big(e(f(x), f(y))/d(x,y)\big) \cdot d(x,y) \leq \big(\mathrm{Diam}_e(f(X))/\ell(\mathcal{P})\big) \cdot d(x,y)$. Suppose that $d(x,y) < \ell(\mathcal{P})$ or, equivalently, $y \in \mathcal{B}_d(x, \ell(\mathcal{P}))$. Then, being $\ell(\mathcal{P})$ a Lebesgue number of $\mathcal{P}$, both x and y are contained in P_n for some n. It follows that $e(f(x), f(y)) \leq M \cdot d(x,y)$. This completes the proof. □

Let $\mathcal{P} = \{P_n\}_{n=1}^k$ be a Lebesgue uniform covering of X and let $\mathcal{Q} = \{Q_n\}_{n=1}^k$ be a family of subsets of X. We say that $\mathcal{Q}$ is a *shrinkage of* $\mathcal{P}$ if it is a Lebesgue uniform covering of X and, for each $n \in \{1, \ldots, k\}$, the closure $\overline{Q_n}$ of Q_n in X is contained in P_n.

Lemma 2.4 *Let (X,d) be a metric space and let $\mathcal{P} = \{P_n\}_{n=1}^k$ be a Lebesgue uniform open covering of X. Then, for each $\delta \in \mathbb{R}^+_*$, there exists a shrinkage $\mathcal{Q}$ of $\mathcal{P}$ such that*

$$\sup_{x\in X} \big|\mathcal{L}(\mathcal{Q})(x) - \mathcal{L}(\mathcal{P})(x)\big| < \delta.$$

Proof. Let C be a non–empty closed subset of X and, for each $t \in \mathbb{R}^+_*$, let $\mathcal{D}_d(C,t) := \{x \in X \,|\, d_C(x) \leq t\}$. Let us prove that, for each $t \in \mathbb{R}^+_*$ and for each $x \in X$, it holds:

$$0 \leq d_C(x) - d_{\mathcal{D}_d(C,t)}(x) \leq t. \tag{12}$$

Fix $t \in \mathbb{R}^+_*$ and $x \in X$. Since $\mathcal{D}_d(C,t)$ contains C, it is evident that $0 \leq d_C(x) - d_{\mathcal{D}_d(C,t)}(x)$. Let $\varepsilon \in \mathbb{R}^+_*$. By definitions of $d_C(x)$, of $\mathcal{D}_d(C,t)$ and of $d_{\mathcal{D}_d(C,t)}(x)$, there exist $y \in \mathcal{D}_d(C,t)$ and $z \in C$ such that

$$d(x,y) \leq d_{\mathcal{D}_d(C,t)}(x) + \varepsilon \quad \text{and} \quad d(y,z) \leq d_C(y) + \varepsilon \leq t + \varepsilon.$$

In particular, we have that $d_C(x) \leq d(x,z) \leq d(x,y) + d(y,z) \leq d_{\mathcal{D}_d(C,t)}(x) + t + 2\varepsilon$ for each $\varepsilon \in \mathbb{R}^+_*$. This implies (12).

Let us complete the proof. For each $n \in \{1, \ldots, k\}$, choose a point x_n of P_n. Define the positive real number s by setting

$$s := \frac{1}{2} \cdot \min\big\{\delta, \ell(\mathcal{P}), d_{X\setminus P_1}(x_1), d_{X\setminus P_2}(x_2), \ldots, d_{X\setminus P_k}(x_k)\big\}$$

and the family $\mathcal{Q} := \{Q_n\}_{n=1}^k$ of open subsets of X by setting

$$Q_n := X \setminus \mathcal{D}_d(X \setminus P_n, s) = \{x \in X \,|\, d_{X\setminus P_n}(x) > s\}.$$

Each Q_n contains the point x_n (and hence it is non–empty) and its closure in X is contained in $d^{-1}_{X\setminus P_n}([s, +\infty))$ (and hence in P_n). Moreover, thanks to (12), we infer that

$$\begin{aligned} 0 \leq \mathcal{L}(\mathcal{P}) - \mathcal{L}(\mathcal{Q}) &\leq \max\big\{\sup(d_{X\setminus P_1} - d_{\mathcal{D}_d(X\setminus P_1, s)}), \ldots, \sup(d_{X\setminus P_k} - d_{\mathcal{D}_d(X\setminus P_k, s)})\big\} \leq \\ &\leq s \leq \min\left\{\frac{\delta}{2}, \frac{\ell(\mathcal{P})}{2}\right\}. \end{aligned}$$

It follows that $\sup_{x\in X} \big|\mathcal{L}(\mathcal{Q})(x) - \mathcal{L}(\mathcal{P})(x)\big| \leq \delta/2 < \delta$ and $\ell(\mathcal{Q}) \geq \ell(\mathcal{P})/2 > 0$. Since $\ell(\mathcal{Q}) > 0$, $\mathcal{Q}$ is a Lebesgue uniform covering of X and hence it is a desired shrinkage of $\mathcal{P}$. □

A preparatory version of Theorem 1L. Let (X,d) be a metric space, let $f : (X,d) \longrightarrow (Y,e)$ be a Lipschitz map between metric spaces and let S be a non–empty subset of X. If $\mathrm{Diam}_d(S)$ is finite, then $\mathrm{Diam}_e(f(S))$ is also finite. In fact, it holds:

$$\mathrm{Diam}_e(f(S)) \leq \mathrm{Lip}(f) \cdot \mathrm{Diam}_d(S). \tag{13}$$

Equip $X \times \mathbb{R}$ and each of its subsets with the metric D defined in (2).

Let B be a bi–Lipschitz locally collared compact subspace of X and let $(\mathcal{U}, \mathbf{h}) = (\{U_n\}_{n=1}^k, \{h_n : U_n \times [0,1) \longrightarrow \Omega_n\}_{n=1}^k)$ be a system of bi–Lipschitz local collars of B. Since B is compact, its diameter β is finite. It follows that each open subset U_n of B and hence each product $U_n \times [0,1)$ has finite diameter. By (13), each open subset Ω_n of X has also finite diameter. Let $X^* := \bigcup_{n=1}^k \Omega_n$, let $E := \mathrm{Diam}_d(X^*)$ and let $\omega := \max_{n \in \{1,\ldots,k\}} \mathrm{Diam}_d(\Omega_n)$. Consider two points x and y in X^*. Then there exist $a, b \in \{1, \ldots, k\}$ (possibly equal) such that $x \in \Omega_a$ and $y \in \Omega_b$. Choose $x' \in U_a = B \cap \Omega_a$ and $y' \in U_b = B \cap \Omega_b$. We have: $d(x,y) \leq d(x,x') + d(x',y') + d(y',y) \leq \omega + \beta + \omega = \beta + 2\omega$. It follows that E is finite and it holds:

$$E \leq \beta + 2\omega. \tag{14}$$

In order to prove Theorem 1L, we need a technical lemma, whose proof is based on an careful modification of an argument of R. Connelly (see [2]).

Lemma 2.5 *Let (X,d) be a metric space, let B be a bi–Lipschitz locally collared subspace of X, let $(\mathcal{U}, \mathbf{h}) = (\{U_n\}_{n=1}^k, \{h_n : U_n \times [0,1) \longrightarrow \Omega_n\}_{n=1}^k)$ be a Lebesgue uniform system of bi–Lipschitz local collars of B, let $\mathcal{V} = \{V_n\}_{n=1}^k$ be a shrinkage of $\mathcal{U}$ (which exists by Lemma 2.4) and let $X^* := \bigcup_{n=1}^k \Omega_n$. Define the positive real numbers ℓ and r by setting*

$$\ell := \min\{1, \ell(\mathcal{V})\} \quad \textit{and} \quad r := \min\{1, r(\mathcal{V})\}$$

Then, for each $\varepsilon \in (0,1]$, there exists a weakly bi–Lipschitz collar $H_\varepsilon : B \times [0,1) \longrightarrow \Omega_\varepsilon$ of B such that

$$\mathrm{Lip}^*(H_\varepsilon) \leq (\sqrt{2})^{3k+1} \cdot (1+\varepsilon) \cdot (1 + 2\varepsilon + 2\varepsilon^2 r)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(h_n) \tag{15}$$

and

$$\mathrm{Lip}^*(H_\varepsilon^{-1}) \leq (\sqrt{2})^{3k+1} \cdot \max\left\{\frac{1}{\varepsilon\ell}, 1 + \frac{1}{\ell}\right\} \cdot (1 + 2\varepsilon + 2\varepsilon^2 r)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(h_n). \tag{16}$$

Moreover, H_ε extends to a weakly bi–Lipschitz homeomorphism from $B \times [0,1]$ to the closure of Ω_ε in X.

Finally, if X is compact, then, for each $\varepsilon \in (0,1]$, there exists a bi–Lipschitz collar $H_\varepsilon : B \times [0,1) \longrightarrow \Omega_\varepsilon$ of B satisfying inequalities (15) and (16), and the following ones:

$$\mathrm{Lip}(H_\varepsilon) \leq \sqrt{2} \cdot (1+\varepsilon) \cdot \left(\max\left\{(\sqrt{2})^3 u(1 + 2\varepsilon + 2\varepsilon^2 r), \frac{E + \varepsilon r}{\eta}\right\}\right)^k,$$

$$\mathrm{Lip}(H_\varepsilon^{-1}) \leq \sqrt{2} \cdot \max\left\{\frac{1}{\varepsilon\ell}, 1 + \frac{1}{\ell}\right\} \cdot \left(\max\left\{(\sqrt{2})^3 u(1 + 2\varepsilon + 2\varepsilon^2 r), \frac{E + \varepsilon r}{\eta}\right\}\right)^k,$$

where E, η and u are the positive real numbers defined by setting $E := \mathrm{Diam}_d(X^*)$, $\eta := \min_{n \in \{1,\ldots,k\}} d\left(h_n\left(\overline{V_n} \times [0, \frac{1}{2}]\right), X^* \setminus \Omega_n\right)$ and $u := \max_{n \in \{1,\ldots,k\}} \mathrm{biLip}(h_n)$. Moreover, H_ε extends to a bi–Lipschitz homeomorphism from $B \times [0,1]$ to the closure of Ω_ε in X.

Proof. We subdivide the proof into five steps.

Step I. Let $n \in \{1, \ldots, k\}$. Consider the subspaces $U_n^* := U_n \times (-\infty, 1)$ and $O_n := (\Omega_n \times \{0\}) \cup (U_n \times \mathbb{R}^-)$ of the metric space $(X \times \mathbb{R}, D)$. Define the map $T_n : U_n^* \longrightarrow O_n$ by setting $T_n(x,t) := h_n(x,t)$ if $(x,t) \in U_n \times [0,1)$ and $T_n(x,t) := (x,t)$ if $(x,t) \in U_n \times \mathbb{R}^-_*$. Observe that T_n is a homeomorphism and it holds: $T_n^{-1}(y,0) = h_n^{-1}(y)$ if $y \in \Omega_n$ and $T_n^{-1}(y,s) = (y,s)$ if $(y,s) \in U_n \times \mathbb{R}^-_*$.

Let us prove that T_n is weakly bi–Lipschitz and compute explicit upper bounds for $\mathrm{Lip}^*(T_n)$ and $\mathrm{Lip}^*(T_n^{-1})$.

Let $(x_0, t_0) \in U_n^*$. Observe that the restriction of T_n to $U_n^* \setminus (U_n \times \{0\})$ is LIP. If $t_0 > 0$, then $\mathrm{Lip}_{(x_0,t_0)}(T_n) = \mathrm{Lip}_{(x_0,t_0)}(h_n)$. If $t_0 < 0$, then $\mathrm{Lip}_{(x_0,t_0)}(T_n) = 1$. Suppose $t_0 = 0$. Let $r \in \mathbb{R}_*^+$ and let $(x,t),(y,s) \in \mathcal{B}_D((x_0,0),r) \cap U_n^*$. Define $U_n(x_0,r) := \mathcal{B}_D((x_0,0),r) \cap (U_n \times [0,1))$. It $t \geq 0$ and $s \geq 0$, then

$$D(T_n(x,t), T_n(y,s)) \leq \mathrm{Lip}(h_n|_{U_n(x_0,r)}) D((x,t),(y,s)).$$

It $t \leq 0$ and $s < 0$, then $D(T_n(x,t), T_n(y,s)) = D((x,t),(y,s))$. Suppose $t > 0$ and $s \leq 0$. Since $D((x,0),(x_0,0)) = d(x,x_0) \leq D((x,t),(x_0,0)) < r$, we have that $(x,0) \in \mathcal{B}_D((x_0,0),r) \cap U_n^*$. It holds:

$$\begin{aligned} D(T_n(x,t), T_n(y,s)) &\leq D(T_n(x,t), T_n(x,0)) + D(T_n(x,0), T_n(y,s)) \leq \\ &\leq \mathrm{Lip}(h_n|_{U_n(x_0,r)})|t| + d(x,y) + |s| \leq \\ &\leq \max\{1, \mathrm{Lip}(h_n|_{U_n(x_0,r)})\}(|t| + |s|) + d(x,y) \leq \\ &\leq \max\{1, \mathrm{Lip}(h_n|_{U_n(x_0,r)})\}\big(|t-s| + d(x,y)\big) \leq \\ &\leq \sqrt{2} \cdot \max\big\{1, \mathrm{Lip}(h_n|_{U_n(x_0,r)})\big\} \cdot D((x,t),(y,s)). \end{aligned}$$

It follows that $\mathrm{Lip}(T_n|_{\mathcal{B}_D((x_0,0),r)\cap U_n^*}) \leq \sqrt{2} \cdot \max\big\{1, \mathrm{Lip}(h_n|_{U_n(x_0,r)})\big\}$ for each $r \in \mathbb{R}_*^+$ and hence $\mathrm{Lip}_{(x_0,0)}(T_n) \leq \sqrt{2} \cdot \max\big\{1, \mathrm{Lip}_{(x_0,0)}(h_n)\big\}$. We infer that

$$\mathrm{Lip}^*(T_n) \leq \sqrt{2} \cdot \max\{1, \mathrm{Lip}^*(h_n)\}. \tag{17}$$

Let now $(y_0, s_0) \in O_n$. Observe that the restriction of T_n^{-1} to $O_n \setminus (U_n \times \{0\})$ is LIP. If $(y_0, s_0) \in (\Omega_n \setminus B) \times \{0\}$, then $\mathrm{Lip}_{(y_0,s_0)}(T_n^{-1}) = \mathrm{Lip}_{(y_0,s_0)}(h_n^{-1})$. Moreover, if $(y_0,s_0) \in U_n \times \mathbb{R}_*^-$, then $\mathrm{Lip}_{(y_0,s_0)}(T_n^{-1}) = 1$. Let $(y_0,t_0) \in U_n \times \{0\}$. Let $r \in \mathbb{R}_*^+$ and let $(x,t),(y,s) \in \mathcal{B}_D((y_0,0),r) \cap O_n$. If $t = 0$ and $s = 0$, then $x, y \in \mathcal{B}_d(y_0,r) \cap \Omega_n$ and hence

$$D(T_n^{-1}(x,t), T_n^{-1}(y,s)) \leq \mathrm{Lip}(h_n^{-1}|_{\mathcal{B}_d(y_0,r)\cap\Omega_n}) D((x,t),(y,s)).$$

If $t < 0$ and $s < 0$, then $D(T_n^{-1}(x,t), T_n^{-1}(y,s)) = D((x,t),(y,s))$. Suppose $t = 0$ and $s < 0$. Observe that $d(y,y_0) \leq D((y,s),(y_0,0)) < r$ and hence $y \in \mathcal{B}_d(y_0,r) \cap U_n \subset \mathcal{B}_d(y_0,r) \cap \Omega_n$. We have:

$$\begin{aligned} D(T_n^{-1}(x,0), T_n^{-1}(y,s)) &\leq D(T_n^{-1}(x,0), T_n^{-1}(y,0)) + D(T_n^{-1}(y,0), T_n^{-1}(y,s)) \leq \\ &\leq \mathrm{Lip}(h_n^{-1}|_{\mathcal{B}_d(y_0,r)\cap\Omega_n}) d(x,y) + |s| \leq \\ &\leq \sqrt{2} \cdot \max\big\{1, \mathrm{Lip}(h_n^{-1}|_{\mathcal{B}_d(y_0,r)\cap\Omega_n})\big\} \cdot D((x,0),(y,s)). \end{aligned}$$

It follows that $\mathrm{Lip}(T_n^{-1}|_{\mathcal{B}_D((y_0,0),r)\cap O_n}) \leq \sqrt{2} \cdot \max\big\{1, \mathrm{Lip}(h_n^{-1}|_{\mathcal{B}_d(y_0,r)\cap\Omega_n})\big\}$ for each $r \in \mathbb{R}_*^+$ and hence $\mathrm{Lip}_{(y_0,0)}(T_n^{-1}) \leq \sqrt{2} \cdot \max\big\{1, \mathrm{Lip}_{y_0}(h_n^{-1})\big\}$. We infer that

$$\mathrm{Lip}^*(T_n^{-1}) \leq \sqrt{2} \cdot \max\big\{1, \mathrm{Lip}^*(h_n^{-1})\big\}. \tag{18}$$

We have just proved that T_n is weakly bi–Lipschitz and it satisfies inequalities (17) and (18). Proceeding in a similar way, we obtain that T_n is also bi–Lipschitz and it hold:

$$\mathrm{Lip}(T_n) \leq \sqrt{2} \cdot \max\{1, \mathrm{Lip}(h_n)\}, \tag{19}$$

$$\mathrm{Lip}(T_n^{-1}) \leq \sqrt{2} \cdot \max\big\{1, \mathrm{Lip}(h_n^{-1})\big\}. \tag{20}$$

Step II. Fix $\varepsilon \in (0,1]$. Denote by $f_{\varepsilon,0} : B \longrightarrow \mathbb{R}$ the null function on B and, for each $n \in \{1, \ldots, k\}$, define the function $f_{\varepsilon,n} : B \longrightarrow \mathbb{R}$ by setting

$$f_{\varepsilon,n}(x) := -\varepsilon \cdot \min\big\{1, \max\{d_{B\setminus V_1}(x), \ldots, d_{B\setminus V_n}(x)\}\big\}$$

for each $x \in B$. It is immediate to verify that

$$\sup_{x\in B} |f_{\varepsilon,n}(x)| \leq \varepsilon r \quad \text{for each } n \in \{1,\ldots,k\} \tag{21}$$

and

$$\inf_{x\in B} |f_{\varepsilon,k}(x)| = \varepsilon \ell. \tag{22}$$

By a simple inductive argument on n, one obtains:

$$\mathrm{Lip}(f_{\varepsilon,n}) \leq \varepsilon \quad \text{for each } n \in \{0,1,\ldots,k\}. \tag{23}$$

For each $n \in \{0,1,\ldots,k\}$, define the subspaces $P_{\varepsilon,n}$ and $Q_{\varepsilon,n}$ of U_n^* by setting

$$P_{\varepsilon,n} := \{(x,t) \in U_n^* \,|\, t \geq f_{\varepsilon,n-1}(x)\} \quad \text{and} \quad Q_{\varepsilon,n} := \{(x,t) \in U_n^* \,|\, t \geq f_{\varepsilon,n}(x)\},$$

and the map $S_{\varepsilon,n} : P_{\varepsilon,n} \longrightarrow Q_{\varepsilon,n}$ as follows: $S_{\varepsilon,n}(x,t) := (x,t)$ if $(x,t) \in P_{\varepsilon,n} \cap \big(U_n \times (\frac{1}{2}, 1)\big)$ and

$$S_{\varepsilon,n}(x,t) := \left(x, \left(\frac{f_{\varepsilon,n}(x) - \frac{1}{2}}{f_{\varepsilon,n-1}(x) - \frac{1}{2}}\right) \cdot \left(t - \frac{1}{2}\right) + \frac{1}{2}\right)$$

if $(x,t) \in P_{\varepsilon,n} \cap \big(U_n \times (-\infty, \frac{1}{2}]\big)$. Let $n \in \{1,\ldots,k\}$. The map $S_{\varepsilon,n}|_{P_{\varepsilon,n}\cap(U_n\times(-\infty,\frac{1}{2}])}$ has an elementary affine interpretation: for each $x \in U_n$, the function from the interval $[f_{\varepsilon,n-1}(x), \frac{1}{2}]$ to the interval $[f_{\varepsilon,n}(x), \frac{1}{2}]$, which sends $t \in [f_{\varepsilon,n-1}(x), \frac{1}{2}]$ into $S_{\varepsilon,n}(x,t) \in [f_{\varepsilon,n}(x), \frac{1}{2}]$, is the restriction of the unique affine function from $\mathbb{R}$ to $\mathbb{R}$, which sends $\frac{1}{2}$ into itself and $f_{\varepsilon,n-1}(x)$ in $f_{\varepsilon,n}(x)$. In this way, it turns out to be evident that the map $S_{\varepsilon,n}$ is a homeomorphism. Moreover, for each $(y,s) \in Q_{\varepsilon,n}$, $S_{\varepsilon,n}^{-1}(y,s) = (y,s)$ if $s > \frac{1}{2}$ and

$$S_{\varepsilon,n}^{-1}(y,s) := \left(y, \left(\frac{f_{\varepsilon,n-1}(y) - \frac{1}{2}}{f_{\varepsilon,n}(y) - \frac{1}{2}}\right) \cdot \left(s - \frac{1}{2}\right) + \frac{1}{2}\right).$$

if $(y,s) \in Q_{\varepsilon,n} \cap \big(U_n \times (-\infty, \frac{1}{2}]\big)$.

Let us prove that $S_{\varepsilon,n}$ is bi–Lipschitz and compute explicit upper bounds for $\mathrm{Lip}(S_{\varepsilon,n})$ and $\mathrm{Lip}(S_{\varepsilon,n}^{-1})$.

Denote by $F_{\varepsilon,n} : U_n \times U_n \longrightarrow \mathbb{R}^+$ the function, which sends $(x,y) \in U_n \times U_n$ into $(f_{\varepsilon,n}(x) - 1/2)(f_{\varepsilon,n-1}(y) - 1/2)^{-1} \in \mathbb{R}^+$. By (21), it follows that

$$\sup_{x\in U_n} F_{\varepsilon,n}(x,x) \leq 1 + 2\varepsilon r. \tag{24}$$

Let $x, y \in U_n$. Since we have that

$$\begin{aligned}
\left(\tfrac{1}{2} - f_{\varepsilon,n-1}(x)\right)\left|F_{\varepsilon,n}(x,x) - F_{\varepsilon,n}(y,y)\right| &\leq \left(\tfrac{1}{2} - f_{\varepsilon,n-1}(x)\right)\left(\left|F_{\varepsilon,n}(x,x) - F_{\varepsilon,n}(y,x)\right| + \right.\\
&\quad \left. + \left|F_{\varepsilon,n}(y,x) - F_{\varepsilon,n}(y,y)\right|\right) \leq \\
&\leq \left|f_{\varepsilon,n}(x) - f_{\varepsilon,n}(y)\right| + \\
&\quad + F_{\varepsilon,n}(y,y) \cdot \left|f_{\varepsilon,n-1}(x) - f_{\varepsilon,n-1}(y)\right|,
\end{aligned}$$

(23) and (24) imply that

$$\left(\tfrac{1}{2} - f_{\varepsilon,n-1}(x)\right)\left|F_{\varepsilon,n}(x,x) - F_{\varepsilon,n}(y,y)\right| \leq (2\varepsilon + 2\varepsilon^2 r)d(x,y). \tag{25}$$

Let now $(x,t),(y,s) \in P_{\varepsilon,n}$. If $t,s > \frac{1}{2}$, then $D(S_{\varepsilon,n}(x,t), S_{\varepsilon,n}(y,s)) = D((x,t),(y,s))$. Suppose $t,s \leq \frac{1}{2}$. Define $R := D(S_{\varepsilon,n}(x,t), S_{\varepsilon,n}(y,s))$. Bearing in mind (24) and (25), we infer that

$$\begin{aligned}
R &\leq d(x,y) + |F_{\varepsilon,n}(x,x) \cdot (t - 1/2) - F_{\varepsilon,n}(y,y) \cdot (s - 1/2)| \leq \\
&\leq d(x,y) + |t - 1/2| \cdot \left|F_{\varepsilon,n}(x,x) - F_{\varepsilon,n}(y,y)\right| + F_{\varepsilon,n}(y,y) \cdot |t - s| \leq \\
&\leq d(x,y) + \left(\tfrac{1}{2} - f_{\varepsilon,n-1}(x)\right)\left|F_{\varepsilon,n}(x,x) - F_{\varepsilon,n}(y,y)\right| + (1 + 2\varepsilon r)|t - s| \leq \\
&\leq (1 + 2\varepsilon + 2\varepsilon^2 r)d(x,y) + (1 + 2\varepsilon r)|t - s|
\end{aligned}$$

and hence

$$D(S_{\varepsilon,n}(x,t), S_{\varepsilon,n}(y,s)) \leq (1 + 2\varepsilon + 2\varepsilon^2 r)(d(x,y) + |t-s|). \tag{26}$$

In particular, we have:

$$D(S_{\varepsilon,n}(x,t), S_{\varepsilon,n}(y,s)) \leq \sqrt{2}(1 + 2\varepsilon + 2\varepsilon^2 r) D((x,t),(y,s)).$$

Let $t > \frac{1}{2}$ and $s \leq \frac{1}{2}$. Thanks to (26), we obtain:

$$\begin{aligned} D(S_{\varepsilon,n}(x,t), S_{\varepsilon,n}(y,s)) &\leq D(S_{\varepsilon,n}(x,t), S_{\varepsilon,n}(x,\tfrac{1}{2})) + D(S_{\varepsilon,n}(x,\tfrac{1}{2}), S_{\varepsilon,n}(y,s)) \leq \\ &\leq |t - 1/2| + (1 + 2\varepsilon + 2\varepsilon^2 r)(d(x,y) + |1/2 - s|) \leq \\ &\leq \sqrt{2}(1 + 2\varepsilon + 2\varepsilon^2 r) D((x,t),(y,s)). \end{aligned}$$

It follows that

$$\mathrm{Lip}(S_{\varepsilon,n}) \leq \sqrt{2}(1 + 2\varepsilon + 2\varepsilon^2 r).$$

In a similar way, one can prove that the preceding inequality remains valid if we replace $\mathrm{Lip}(S_{\varepsilon,n})$ with $\mathrm{Lip}(S_{\varepsilon,n}^{-1})$. In other words, we have that $S_{\varepsilon,n}$ is bi–Lipschitz and it holds:

$$\max\left\{\mathrm{Lip}(S_{\varepsilon,n}), \mathrm{Lip}(S_{\varepsilon,n}^{-1})\right\} \leq \sqrt{2}(1 + 2\varepsilon + 2\varepsilon^2 r). \tag{27}$$

In particular, we infer that

$$\max\left\{\mathrm{Lip}^*(S_{\varepsilon,n}), \mathrm{Lip}^*(S_{\varepsilon,n}^{-1})\right\} \leq \sqrt{2}(1 + 2\varepsilon + 2\varepsilon^2 r). \tag{28}$$

Step III. For each $n \in \{0, 1, \ldots, k\}$, let $X_{\varepsilon,n}$ be the subspace of $X \times \mathbb{R}$ defined by

$$X_{\varepsilon,n} := (X^* \times \{0\}) \cup \{(x,t) \in B \times \mathbb{R}^- \mid t \geq f_{\varepsilon,n}(x)\}.$$

Let $n \in \{1, \ldots, k\}$. Denote by $L_{\varepsilon,n}$ the open subset $X_{\varepsilon,n-1} \cap O_n$ of $X_{\varepsilon,n-1}$, by $O_{\varepsilon,n}$ the open subset $X_{\varepsilon,n} \cap O_n$ of $X_{\varepsilon,n}$ and by $\Phi_{\varepsilon,n} : X_{\varepsilon,n-1} \longrightarrow X_{\varepsilon,n}$ the map defined as follows: $\Phi_{\varepsilon,n}(x,t) := (x,t)$ if $(x,t) \in X_{\varepsilon,n-1} \setminus L_{\varepsilon,n}$ and

$$\Phi_{\varepsilon,n}(x,t) := T_n(S_{\varepsilon,n}(T_n^{-1}(x,t))) \quad \text{if} \ \ (x,t) \in L_{\varepsilon,n}. \tag{29}$$

The map $\Phi_{\varepsilon,n}$ is a homeomorphism and it holds: $\Phi_{\varepsilon,n}^{-1}(y,s) = (y,s)$ if $(y,s) \in X_{\varepsilon,n} \setminus O_{\varepsilon,n}$ and $\Phi_{\varepsilon,n}^{-1}(y,s) := T_n(S_{\varepsilon,n}^{-1}(T_n^{-1}(y,s)))$ if $(y,s) \in O_{\varepsilon,n}$. Define the closed subset $K_{\varepsilon,n}$ of $X_{\varepsilon,n-1}$ and the closed subset $J_{\varepsilon,n}$ of $X_{\varepsilon,n}$ by setting

$$\begin{aligned} K_{\varepsilon,n} &:= X_{\varepsilon,n-1} \cap \left(h_n(\overline{V_n} \times [0, \tfrac{1}{2}]) \cup (\overline{V_n} \times \mathbb{R}^-)\right), \\ J_{\varepsilon,n} &:= X_{\varepsilon,n} \cap \left(h_n(\overline{V_n} \times [0, \tfrac{1}{2}]) \cup (\overline{V_n} \times \mathbb{R}^-)\right) \end{aligned}$$

Observe that $K_{\varepsilon,n} \subset L_{\varepsilon,n}$ and $J_{\varepsilon,n} \subset O_{\varepsilon,n}$. In particular, $\{L_{\varepsilon,n}, X_{\varepsilon,n-1} \setminus K_{\varepsilon,n}\}$ is an open covering of $X_{\varepsilon,n-1}$ and $\{O_{\varepsilon,n}, X_{\varepsilon,n} \setminus J_{\varepsilon,n}\}$ is an open covering of $X_{\varepsilon,n}$. Thanks to (29) and to the definitions of T_n and of $S_{\varepsilon,n}$, we infer that

$$\Phi_{\varepsilon,n}(x,t) = (x,t) \quad \text{for each} \ \ (x,t) \in X_{\varepsilon,n-1} \setminus K_{\varepsilon,n}. \tag{30}$$

Let $(x,t) \in X_{\varepsilon,n-1}$. If $(x,t) \in X_{\varepsilon,n-1} \setminus K_{\varepsilon,n}$, then (30) ensures that $\mathrm{Lip}_{(x,t)}(\Phi_{\varepsilon,n}) \leq 1$ (it is equal to 0 if (x,t) is an isolated point of $X_{\varepsilon,n-1}$ and equal to 1 otherwise). Suppose $(x,t) \in L_{\varepsilon,n}$. Combining Lemma 2.1(2) and above points (17), (18) and (28), we obtain that

$$\begin{aligned} \mathrm{Lip}_{(x,t)}(\Phi_{\varepsilon,n}) &\leq (\sqrt{2})^3 \cdot \max\{1, \mathrm{Lip}^*(h_n)\} \cdot \max\{1, \mathrm{Lip}^*(h_n^{-1})\} \cdot (1 + 2\varepsilon + 2\varepsilon^2 r) = \\ &= (\sqrt{2})^3 \cdot (1 + 2\varepsilon + 2\varepsilon^2 r) \cdot \mathrm{biLip}^*(h_n). \end{aligned}$$

It follows that $\mathrm{Lip}^*(\Phi_{\varepsilon,n}) \leq (\sqrt{2})^3 \cdot (1 + 2\varepsilon + 2\varepsilon^2 r) \cdot \mathrm{biLip}^*(h_n)$. Replacing $X_{\varepsilon,n-1}$ with $X_{\varepsilon,n}$, $L_{\varepsilon,n}$ with $O_{\varepsilon,n}$ and $K_{\varepsilon,n}$ with $J_{\varepsilon,n}$, one can prove, in a similar way, that the latter inequality is true replacing $\mathrm{Lip}^*(\Phi_{\varepsilon,n})$ with $\mathrm{Lip}^*(\Phi_{\varepsilon,n}^{-1})$. It follows that $\Phi_{\varepsilon,n}$ is weakly bi–Lipschitz and it holds:

$$\max\left\{\mathrm{Lip}^*(\Phi_{\varepsilon,n}), \mathrm{Lip}^*(\Phi_{\varepsilon,n}^{-1})\right\} \leq (\sqrt{2})^3 \cdot (1 + 2\varepsilon + 2\varepsilon^2 r) \cdot \mathrm{biLip}^*(h_n). \tag{31}$$

We conclude this step assuming that X is compact and proving that, under this additional condition, $\Phi_{\varepsilon,n}$ is bi–Lipschitz. First, we observe that, by (21), $\mathrm{Diam}_D(X_{\varepsilon,n}) \leq E + \varepsilon r$. Now we need lower bounds for the distances $D(K_{\varepsilon,n}, X_{\varepsilon,n-1} \setminus L_{\varepsilon,n})$ and $D(J_{\varepsilon,n}, X_{\varepsilon,n} \setminus O_{\varepsilon,n})$. Let $(x,t) \in K_{\varepsilon,n}$ and let $(y,s) \in X_{\varepsilon,n-1} \setminus L_{\varepsilon,n}$. If $t = 0$, then $x \in h_n(\overline{V_n} \times [0, \frac{1}{2}])$. If $t < 0$, then $x \in \overline{V_n} \subset h_n(\overline{V_n} \times [0, \frac{1}{2}])$. If $s = 0$, then $y \in X^* \setminus \Omega_n$. Finally, if $s < 0$, then $y \in B \setminus U_n \subset X^* \setminus \Omega_n$. In any case, $x \in h_n(\overline{V_n} \times [0, \frac{1}{2}])$, $y \in X^* \setminus \Omega_n$ and hence $D((x,t),(y,s)) \geq d(x,y) \geq \eta$. It follows that

$$D(K_{\varepsilon,n}, X_{\varepsilon,n-1} \setminus L_{\varepsilon,n}) \geq \eta. \tag{32}$$

Proceeding similarly, one can prove that

$$D(J_{\varepsilon,n}, X_{\varepsilon,n} \setminus O_{\varepsilon,n}) \geq \eta. \tag{33}$$

Combining Lemma 2.1(1), Lemma 2.3(1), (19), (20), (27) and (32), we obtain that

$$\mathrm{Lip}(\Phi_{\varepsilon,n}) \leq \max\left\{(\sqrt{2})^3 u(1 + 2\varepsilon + 2\varepsilon^2 r), \frac{E + \varepsilon r}{\eta}\right\}. \tag{34}$$

Using (33) instead of (32), one obtain also:

$$\mathrm{Lip}(\Phi_{\varepsilon,n}^{-1}) \leq \max\left\{(\sqrt{2})^3 u(1 + 2\varepsilon + 2\varepsilon^2 r), \frac{E + \varepsilon r}{\eta}\right\}. \tag{35}$$

Step IV. Identify X^* with $X^* \times \{0\}$. Define $\Phi_\varepsilon : X^* = X_{\varepsilon,0} \longrightarrow X_{\varepsilon,k}$ by setting $\Phi_\varepsilon := \Phi_{\varepsilon,k} \circ \Phi_{\varepsilon,k-1} \circ \ldots \circ \Phi_{\varepsilon,2} \circ \Phi_{\varepsilon,1}$. The map Φ_ε is a homeomorphism and its inverse is equal to $\Phi_{\varepsilon,1}^{-1} \circ \Phi_{\varepsilon,2}^{-1} \circ \ldots \circ \Phi_{\varepsilon,k-1}^{-1} \circ \Phi_{\varepsilon,k}^{-1}$. Combining Lemma 2.1(2) with (31), we obtain that Φ_ε is weakly bi–Lipschitz and it holds:

$$\max\left\{\mathrm{Lip}^*(\Phi_\varepsilon), \mathrm{Lip}^*(\Phi_\varepsilon^{-1})\right\} \leq (\sqrt{2})^{3k} \cdot (1 + 2\varepsilon r + 2\varepsilon^2 r)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(h_n). \tag{36}$$

If B is compact, then Lemma 2.1(1) and points (34) and (35) imply that Φ_ε is bi–Lipschitz and it holds:

$$\max\left\{\mathrm{Lip}(\Phi_\varepsilon), \mathrm{Lip}(\Phi_\varepsilon^{-1})\right\} \leq \left(\max\left\{(\sqrt{2})^3 u(1 + 2\varepsilon r + 2\varepsilon^2 r), \frac{E + \varepsilon r}{\eta}\right\}\right)^k. \tag{37}$$

Step V. Let $X^+ := (X^* \times \{0\}) \cup (B \times [0,1])$ and let $\Psi_\varepsilon : X_{\varepsilon,k} \longrightarrow X^+$ be the map, which sends $(x,0) \in X_{\varepsilon,k}$ into $(x,0) \in X^+$ and $(x,t) \in X_{\varepsilon,k}$ with $t < 0$ into $(x, t/f_{\varepsilon,k}(x)) \in X^+$. The latter map is a homeomorphism and it holds: $\Psi_\varepsilon^{-1}(y,0) = (y,0)$ if $(y,0) \in X^+$ and $\Psi_\varepsilon^{-1}(y,s) = (y, s f_{\varepsilon,k}(y))$ if $(y,s) \in X^+$ and $s > 0$.

Let us prove that Ψ_ε is bi–Lipschitz and compute explicit upper bounds for $\mathrm{Lip}(\Phi_\varepsilon)$ and $\mathrm{Lip}(\Phi_\varepsilon^{-1})$.

Let $(x,t),(y,s) \in X_{\varepsilon,k}$. If $t = s = 0$, then

$$D(\Psi_\varepsilon(x,t), \Psi_\varepsilon(y,s)) = D((x,t),(y,s)). \tag{38}$$

Suppose $t, s < 0$. By (22) and (23), we obtain:

$$\begin{aligned} D(\Psi_\varepsilon(x,t), \Psi_\varepsilon(y,s)) &\leq d(x,y) + \left| \frac{t}{f_{\varepsilon,k}(x)} - \frac{s}{f_{\varepsilon,k}(y)} \right| \leq \\ &\leq d(x,y) + \frac{1}{|f_{\varepsilon,k}(x)|} \cdot |t-s| + |s| \cdot \left| \frac{1}{f_{\varepsilon,k}(x)} - \frac{1}{f_{\varepsilon,k}(y)} \right| \leq \\ &\leq d(x,y) + \frac{1}{\varepsilon\ell} \cdot |t-s| + |f_{\varepsilon,k}(y)| \cdot \left| \frac{f_{\varepsilon,k}(y) - f_{\varepsilon,k}(x)}{f_{\varepsilon,k}(x) f_{\varepsilon,k}(y)} \right| \leq \\ &\leq d(x,y) + \frac{1}{\varepsilon\ell} \cdot |t-s| + \frac{1}{\varepsilon\ell} \cdot \varepsilon \cdot d(x,y) = \\ &= \frac{1}{\varepsilon\ell} \cdot |t-s| + \left(1 + \frac{1}{\ell}\right) \cdot d(x,y). \end{aligned}$$

and hence

$$D(\Psi_\varepsilon(x,t), \Psi_\varepsilon(y,s)) \leq \sqrt{2} \cdot \max\left\{ \frac{1}{\varepsilon\ell}, 1 + \frac{1}{\ell} \right\} \cdot D((x,t),(y,s)). \tag{39}$$

If $t = 0$ and $s < 0$, then it holds:

$$\begin{aligned} D(\Psi_\varepsilon(x,0), \Psi_\varepsilon(y,s)) &\leq D(\Psi_\varepsilon(x,0), \Psi_\varepsilon(y,0)) + D(\Psi_\varepsilon(y,0), \Psi_\varepsilon(y,s)) = \\ &= d(x,y) + \frac{|s|}{|f_{\varepsilon,k}(y)|} \leq d(x,y) + \frac{1}{\varepsilon\ell} \cdot |s| \leq \\ &\leq \sqrt{2} \cdot \frac{1}{\varepsilon\ell} \cdot D((x,0),(y,s)). \end{aligned} \tag{40}$$

Combining (38), (39) and (40), we obtain:

$$\operatorname{Lip}(\Psi_\varepsilon) \leq \sqrt{2} \cdot \max\left\{ \frac{1}{\varepsilon\ell}, 1 + \frac{1}{\ell} \right\}. \tag{41}$$

In particular, we have:

$$\operatorname{Lip}^*(\Psi_\varepsilon) \leq \sqrt{2} \cdot \max\left\{ \frac{1}{\varepsilon\ell}, 1 + \frac{1}{\ell} \right\}. \tag{42}$$

Let now $(x,t), (y,s) \in X^+$. If $t = s = 0$, then $D(\Psi_\varepsilon^{-1}(x,t), \Psi_\varepsilon^{-1}(y,s)) = D((x,t),(y,s))$. Suppose $t, s > 0$. Thanks to (22) and (23), we obtain:

$$\begin{aligned} D(\Psi_\varepsilon^{-1}(x,t), \Psi_\varepsilon^{-1}(y,s)) &\leq d(x,y) + \left| t f_{\varepsilon,k}(x) - s f_{\varepsilon,k}(y) \right| \leq \\ &\leq d(x,y) + |f_{\varepsilon,k}(x)| \cdot |t-s| + s \cdot \left| f_{\varepsilon,k}(x) - f_{\varepsilon,k}(y) \right| \leq \\ &\leq d(x,y) + \varepsilon r |t-s| + \varepsilon \cdot d(x,y) = \\ &= (1+\varepsilon) d(x,y) + \varepsilon r |t-s| \leq \\ &\leq \sqrt{2} \cdot (1+\varepsilon) \cdot D((x,t),(y,s)). \end{aligned}$$

If $t = 0$ and $s > 0$, then it holds:

$$\begin{aligned} D(\Psi_\varepsilon^{-1}(x,0), \Psi_\varepsilon^{-1}(y,s)) &\leq D(\Psi_\varepsilon^{-1}(x,0), \Psi_\varepsilon^{-1}(y,0)) + D(\Psi_\varepsilon^{-1}(y,0), \Psi_\varepsilon^{-1}(y,s)) \leq \\ &\leq d(x,y) + |s| \cdot |f_{\varepsilon,k}(y)| \leq d(x,y) + \varepsilon r |s| \leq \\ &\leq \sqrt{2} \cdot D((x,t),(y,s)). \end{aligned}$$

It follows that

$$\operatorname{Lip}(\Psi_\varepsilon^{-1}) \leq \sqrt{2}(1+\varepsilon). \tag{43}$$

In particular, we have:

$$\mathrm{Lip}^*(\Psi_\varepsilon^{-1}) \leq \sqrt{2}(1+\varepsilon). \tag{44}$$

Finally, let $\Theta_\varepsilon : X^* \longrightarrow X^+$ be the weakly bi–Lipschitz map defined as the composition $\Psi_\varepsilon \circ \Phi_\varepsilon$, let $\Omega_\varepsilon := (\Theta_\varepsilon)^{-1}(B \times (0,1])$ and let $\Theta_\varepsilon^* : \Omega_\varepsilon \longrightarrow B \times (0,1]$ be the weakly bi–Lipschitz map defined as the restriction of Θ_ε from Ω_ε to $B \times (0,1]$. Define $H_\varepsilon : B \times [0,1) \longrightarrow \Omega_\varepsilon$ by setting

$$H_\varepsilon(x,t) := \left(\Theta_\varepsilon^*\right)^{-1}(x, 1-t)$$

for each $(x,t) \in B \times [0,1)$. Such a map is a weakly bi–Lipschitz collar of B and, thanks to Lemma 2.1(2) and to points (36), (42) and (44), we obtain the first two upper bounds in the statement of the lemma. Evidently, H_ε admits a weakly bi–Lipschitz homeomorphic extension from $B \times [0,1]$ to $\overline{\Omega_\varepsilon} = (\Theta_\varepsilon)^{-1}(B \times [0,1])$. If B is compact, then Lemma 2.1(1) and points (37), (41) and (43) imply the remaining part of the lemma. □

Remark 2.6 Thanks to inequality (14), in the statement of Lemma 2.5, one may replace E with $\beta + 2\omega$, where $\beta := \mathrm{Diam}_d(B)$ and $\omega := \max_{n \in \{1,\ldots,k\}} \mathrm{Diam}_d(\Omega_n)$.

Proof of Theorems 1L and 1L*. Let (X,d), B, $(\mathcal{U}, \mathbf{h})$, L and R be as in the statement of Theorem 1L. Define the continuous functions $F : (0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $G : (0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ by setting

$$F(\varepsilon, r) := (\sqrt{2})^{3k+1} \cdot (1+\varepsilon) \cdot (1 + 2\varepsilon + 2\varepsilon^2 r)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(h_n),$$
$$G(\varepsilon, \ell, r) := (\sqrt{2})^{3k+1} \cdot (1+\ell^{-1}) \cdot (1 + 2\varepsilon + 2\varepsilon^2 r)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(h_n).$$

Choose a real number ε' such that

$$(1+L)^{-1} < \varepsilon' < 1.$$

Since $\varepsilon' < 1$, $F(\varepsilon', R) < F(1,R)$ and $G(\varepsilon', L, R) < G(1, L, R)$. By Lemma 2.4, one can choose a shrinkage $\mathcal{V}$ of $\mathcal{U}$ with $\mathcal{L}(\mathcal{V})$ arbitrarily close to $\mathcal{L}(\mathcal{U})$ in the sense of the topology of uniform convergence on B. In particular, $\ell_\mathcal{V} := \min\{1, \ell(\mathcal{V})\}$ is arbitrarily close to L and $r_\mathcal{V} := \min\{1, r(\mathcal{V})\}$ is arbitrarily close to R. In this way, there exists a shrinkage $\mathcal{V}$ of $\mathcal{U}$ such that $F(\varepsilon', r_\mathcal{V}) \leq F(1,R)$, $G(\varepsilon', \ell_\mathcal{V}, r_\mathcal{V}) \leq G(1,L,R)$ and $(1+\ell_\mathcal{V})^{-1} \leq \varepsilon'$. Observe that the latter inequality is equivalent to asserts that $\max\{(\varepsilon \ell_\mathcal{V})^{-1}, 1 + \ell_\mathcal{V}^{-1}\} = 1 + \ell_\mathcal{V}^{-1}$. The weakly bi–Lipschitz collar $H_{\varepsilon'} : B \times [0,1) \longrightarrow \Omega_{\varepsilon'}$ of B found in the first part of Lemma 2.5 has the desired properties. Using the second part of Lemma 2.5, one obtains Theorem 1L*. □

Proof of Theorems 2L and 2L*. Let $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^{m-1}$ be the projection onto the first $m-1$ coordinates and, for each $a \in \{m-1, m\}$, let $\|v\|_a$ be the usual euclidean norm of a vector v of $\mathbb{R}^a$. For each $n \in \{1, \ldots, k\}$, denote by $\xi_n : U_n \times [0,1) \longrightarrow V_n \times [0,1)$ the homeomorphism sending $(x,t) \in U_n \times [0,1)$ into $(\pi(\phi_n(x)), t) \in V_n \times [0,1)$ and define the homeomorphism $h_n : U_n \times [0,1) \longrightarrow \Omega_n$ by setting $h_n := \phi_n^{-1} \circ \xi_n$.

Let us prove that ξ_n (and hence h_n) is a bi–Lipschitz homeomorphism. Let $(x,t), (y,s) \in U_n \times [0,1)$. It holds:

$$\begin{aligned}\|\xi_n(x,t) - \xi_n(y,s)\|_m &\leq \|\phi_n(x) - \phi_n(y)\|_m + |t-s| \leq \mathrm{Lip}(\phi_n) d(x,y) + |t-s| \leq \\ &\leq \sqrt{2} \cdot \max\{1, \mathrm{Lip}(\phi_n)\} \cdot D((x,t),(y,s)),\end{aligned}$$

where d is the metric of X and D is the usual metric of $X \times \mathbb{R}$ induced by d. It follows that ξ_n is Lipschitz and $\mathrm{Lip}(\xi_n) \leq \sqrt{2} \cdot \max\{1, \mathrm{Lip}(\phi_n)\}$. Let now $(p,t), (q,s) \in V_n \times [0,1)$. It holds:

$$\begin{aligned}D(\xi_n^{-1}(p,t), \xi_n^{-1}(q,s)) &\leq d(\phi_n^{-1}(p,0), \phi_n^{-1}(q,0)) + |t-s| \leq \\ &\leq \mathrm{Lip}(\phi_n^{-1}) \|p-q\|_{m-1} + |t-s| \leq \\ &\leq \sqrt{2} \cdot \max\{1, \mathrm{Lip}(\phi_n^{-1})\} \cdot \|(p,t)-(q,s)\|_m.\end{aligned}$$

It follows that ξ_n^{-1} is Lipschitz and $\mathrm{Lip}(\xi_n^{-1}) \le \sqrt{2} \cdot \max\{1, \mathrm{Lip}(\phi_n^{-1})\}$. By Lemma 2.1(1), we infer that h_n is bi–Lipschitz and it hold:

$$\mathrm{Lip}(h_n) \le \mathrm{Lip}(\phi_n^{-1}) \cdot \sqrt{2} \cdot \max\{1, \mathrm{Lip}(\phi_n)\} \le \sqrt{2} \cdot \mathrm{biLip}(\phi_n),$$
$$\mathrm{Lip}(h_n^{-1}) \le \sqrt{2} \cdot \max\{1, \mathrm{Lip}(\phi_n^{-1})\} \cdot \mathrm{Lip}(\phi_n) \le \sqrt{2} \cdot \mathrm{biLip}(\phi_n)$$

and hence $\mathrm{biLip}(h_n) \le 2\big(\mathrm{biLip}(\phi_n)\big)^2$. Proceeding similarly, we obtain also that

$$\mathrm{biLip}^*(h_n) \le 2\big(\mathrm{biLip}^*(\phi_n)\big)^2. \tag{45}$$

We have just proved that $(\mathcal{U}, \{h_n\}_{n=1}^k)$ is a Lebesgue uniform system of bi–Lipschitz local collars of B. The latter fact, (45) and Theorem 1L (resp. Theorem 1L*) imply immediately Theorem 2L (resp. Theorem 2L*). □

Proof of Theorems 3L and 3L*. We subdivide the proof into three steps.

Step I. By hypothesis of Theorem 3L, $(\mathcal{U}, \mathbf{g}) = (\{U_n\}_{n=1}^k, \{g_n : U_n \times (-1,1) \longrightarrow \Omega_n\}_{n=1}^k)$ is two–sided coherent, so there exists a connected open neighborhood Z of B in X such that each Ω_n is contained in Z and $Z \setminus B$ has exactly two connected components, which we denote by Z^+ and Z^-. Let $n \in \{1, \ldots, k\}$ and let U_n^* be a connected component of U_n. If $g_n(U_n^* \times (0,1)) \subset Z^-$ (and hence $g_n(U_n^* \times (-1,0)) \subset Z^+$), then we modify g_n on $U_n^* \times (-1,1)$ by considering the map, which sends $(x,t) \in U_n^* \times (-1,1)$ into $g_n(x,-t) \in Z$. Proceeding in this way for each $n \in \{1, \ldots, k\}$ and for each connected component U_n^* of U_n such that $g_n(U_n^* \times (0,1)) \subset Z^-$, we may suppose that each map g_n has the following property: $g_n(U_n \times (0,1)) \subset Z^+$ and $g_n(U_n \times (-1,0)) \subset Z^-$. Define the subspaces X^+ and X^- of the metric space X by $X^+ := Z^+ \cup B$ and $X^- := Z^- \cup B$. For each $n \in \{1, \ldots, k\}$, let $\Omega_n^+ := \Omega_n \cap X^+$, let $\Omega_n^- := \Omega_n \cap X^-$ and let $g_n^+ : U_n \times [0,1) \longrightarrow \Omega_n^+$ and $g_n^- : U_n \times [0,1) \longrightarrow \Omega_n^-$ be the bi–Lipschitz homeomorphisms defined by setting

$$h_n^+(x,t) := g_n(x,t) \quad \text{and} \quad h_n^-(x,t) := g_n(x,-t)$$

for each $(x,t) \in U_n \times [0,1)$. The pair $(\mathcal{U}, \{h_n^+\}_{n=1}^k)$ is a Lebesgue uniform system of bi–Lipschitz local collars of B, viewed as a subspace of X^+. Similarly, the pair $(\mathcal{U}, \{h_n^-\}_{n=1}^k)$ is a Lebesgue uniform system of bi–Lipschitz local collars of B, viewed as a subspace of X^-. Observe that, for each $n \in \{1, \ldots, k\}$, $\max\{\mathrm{biLip}^*(h_n^+), \mathrm{biLip}^*(h_n^-)\} \le \mathrm{biLip}^*(g_n)$. Applying Theorem 1L twice, we obtain a closed neighborhood Ξ^+ of B in X^+, a closed neighborhood Ξ^- of B in X^- and two weakly bi–Lipschitz homeomorphisms $G^+ : B \times [0,1] \longrightarrow \Xi^+$ and $G^- : B \times [0,1] \longrightarrow \Xi^-$ such that $G^+(b,0) = b = G^-(b,0)$ for each $b \in B$, $G^+(B \times (0,1))$ and $G^-(B \times (0,1))$ are open subsets of X, and it hold:

$$\mathrm{Lip}^*(G^+) \le (\sqrt{2})^{3k+3} \cdot (3+2R)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(g_n), \tag{46}$$
$$\mathrm{Lip}^*((G^+)^{-1}) \le (\sqrt{2})^{3k+1} \cdot (1+L^{-1}) \cdot (3+2R)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(g_n) \tag{47}$$

and

$$\mathrm{Lip}^*(G^-) \le (\sqrt{2})^{3k+3} \cdot (3+2R)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(g_n), \tag{48}$$
$$\mathrm{Lip}^*((G^-)^{-1}) \le (\sqrt{2})^{3k+1} \cdot (1+L^{-1}) \cdot (3+2R)^k \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(g_n) \tag{49}$$

Define the closed neighborhhod Ξ of B in X by setting $\Xi := \Xi^+ \cup \Xi^-$ and the map $G : B \times [-1,1] \longrightarrow \Xi$ by setting $G(x,t) := G^+(x,t)$ if $(x,t) \in B \times [0,1]$ and $G(x,t) := G^-(x,-t)$ if $(x,t) \in B \times [-1,0)$. Observe that the map G is a homeomorphism and it hold: $G^{-1}(y) = (G^+)^{-1}(y)$ if $y \in \Xi^+$ and $G^{-1}(y) = \theta((G^-)^{-1}(y))$ if $y \in \Xi^-$, where θ denotes the bi–Lipschitz homeomorphism from $B \times [-1,1]$ into itself, which sends (x,t) into $(x,-t)$.

Step II. Let us prove that G is a weakly bi–Lipschitz homeomorphism. First, we prove that G is weakly Lipschitz. Let $(x_0,t_0)\in B\times[-1,1]$. Observe that the restriction of G to $B\times([-1,0)\cup(0,1])$ is LIP. If $t_0>0$, then $\mathrm{Lip}_{(x_0,t_0)}(G)=\mathrm{Lip}_{(x_0,t_0)}(G^+)$. If $t_0<0$, then $\mathrm{Lip}_{(x_0,t_0)}(G)=\mathrm{Lip}_{(x_0,t_0)}(G^-)$. Suppose $t_0=0$. Let $r\in\mathbb{R}^+_*$. Define $V_r:=\mathcal{B}_D((x_0,0),r)\cap(B\times[-1,1])$. Let $(x,t),(y,s)\in V_r$ with $t\geq 0$ and $s<0$. Observe that $D((x,0),(x_0,0))=d(x,x_0)\leq D((x,t),(x_0,0))<r$, $D((y,s),(x_0,0))=D((y,-s),(x_0,0))<r$ and hence $\{(x,0),(y,-s)\}\subset V_r$. We have:

$$\begin{aligned}
d(G(x,t),G(y,s)) &\leq d(G^+(x,t),G^+(x,0))+d(G^-(x,0),G^-(y,-s))\leq\\
&\leq \mathrm{Lip}(G^+|_{V_r})|t|+\mathrm{Lip}(G^-|_{V_r})(d(x,y)+|s|)\leq\\
&\leq \max\{\mathrm{Lip}(G^+|_{V_r}),\mathrm{Lip}(G^-|_{V_r})\}\cdot(d(x,y)+|t-s|)\leq\\
&\leq \sqrt{2}\cdot\max\{\mathrm{Lip}(G^+|_{V_r}),\mathrm{Lip}(G^-|_{V_r})\}\cdot D((x,t),(y,s)).
\end{aligned}$$

It follows that G is weakly Lipschitz and $\mathrm{Lip}_{(x_0,0)}(G)\leq\sqrt{2}\cdot\max\{\mathrm{Lip}_{(x_0,0)}(G^+),\mathrm{Lip}_{(x_0,0)}(G^-)\}$. In particular, we have that $\mathrm{Lip}^*(G)\leq\sqrt{2}\cdot\max\{\mathrm{Lip}^*(G^+),\mathrm{Lip}^*(G^-)\}$. Combining the latter inequality with (46) and (48), we infer that

$$\mathrm{Lip}^*(G)\leq(\sqrt{2})^{3k+4}\cdot(3+2R)^k\cdot\textstyle\prod_{n=1}^k \mathrm{biLip}^*(g_n),$$

as desired.

Let us show that G^{-1} is weakly Lipschitz. Let $b\in\Xi$. Observe that the restriction of G^{-1} to $\Xi\setminus B$ is LIP. If $b\in\Xi^+\setminus B$, then $\mathrm{Lip}_b(G^{-1})=\mathrm{Lip}_b((G^+)^{-1})$. If $b\in\Xi^-\setminus B$, then $\mathrm{Lip}_b(G^{-1})=\mathrm{Lip}_b((G^-)^{-1})$. Let $b\in B$ and let $m\in\{1,\dots,k\}$ such that $b\in\Omega_m$. Choose $\delta\in\mathbb{R}^+_*$. By definition of $\mathrm{Lip}_b((G^+)^{-1})$ and $\mathrm{Lip}_b((G^-)^{-1})$, there exists $s_b\in\mathbb{R}^+_*$ such that $\mathcal{B}_d(b,s_b)\subset\Omega_m\cap\Xi$, $(G^+)^{-1}|_{\mathcal{B}_d(b,s_b)\cap\Xi^+}$ and $(G^-)^{-1}|_{\mathcal{B}_d(b,s_b)\cap\Xi^-}$ are Lipschitz, $\mathrm{Lip}((G^+)^{-1}|_{\mathcal{B}_d(b,s_b)\cap\Xi^+})\leq\mathrm{Lip}_b((G^+)^{-1})+\delta$ and $\mathrm{Lip}((G^-)^{-1}|_{\mathcal{B}_d(b,s_b)\cap\Xi^-})\leq\mathrm{Lip}_b((G^-)^{-1})+\delta$. Denote by $\pi_m:U_m\times(-1,1)\longrightarrow U_m$ the natural projection of $U_m\times(-1,1)$ onto the first factor and define the continuous map $\rho_m:\Omega_m\longrightarrow U_m$ as the composition $\pi_m\circ g_m^{-1}$. Observe that $\rho_m(b)=b$ so there exists $r_b\in(0,s_b]$ such that $\rho_m(\mathcal{B}_d(b,r_b))\subset\mathcal{B}_d(b,s_b)$. Let $r\in(0,r_b]$ and let $p,q\in\mathcal{B}_d(b,r)$. If $p,q\in\Xi^+$, then

$$D(G^{-1}(p),G^{-1}(q))\leq\mathrm{Lip}\big((G^+)^{-1}|_{\mathcal{B}_d(b,s_b)\cap\Xi^+}\big)\cdot d(p,q)\leq\big(\mathrm{Lip}_b((G^+)^{-1})+\delta\big)\cdot d(p,q).$$

If $p,q\in\Xi^-$, then

$$D(G^{-1}(p),G^{-1}(q))\leq\mathrm{Lip}\big((G^-)^{-1}|_{\mathcal{B}_d(b,s_b)\cap\Xi^-}\big)\cdot d(p,q)\leq\big(\mathrm{Lip}_b((G^-)^{-1})+\delta\big)\cdot d(p,q).$$

Suppose $p\in\Xi^+$ and $q\in\Xi^-\setminus B$. Let $(x,t):=g_m^{-1}(p)\in U_m\times[0,1]$ and $(y,s)\in g_m^{-1}(q)\in U_m\times[-1,0)$. Observe that $x=\rho_m(p)$ belongs to $\mathcal{B}_d(b,s_b)\cap U_m$. We have:

$$\begin{aligned}
D(G^{-1}(p),G^{-1}(q)) &\leq D((G^+)^{-1}(p),(G^+)^{-1}(x))+D((G^-)^{-1}(x),(G^-)^{-1}(q))\leq\\
&\leq \mathrm{Lip}((G^+)^{-1}|_{\mathcal{B}_d(b,s_b)\cap\Xi^+})\cdot d(p,x)+\mathrm{Lip}((G^-)^{-1}|_{\mathcal{B}_d(b,s_b)\cap\Xi^-})\cdot d(x,q)\leq\\
&\leq \max\big\{\mathrm{Lip}_b((G^+)^{-1})+\delta,\mathrm{Lip}_b((G^-)^{-1})+\delta\big\}\cdot(d(p,x)+d(x,q)),
\end{aligned}$$

$$\begin{aligned}
d(p,x)+d(x,q) &\leq \mathrm{Lip}(g_m)D((x,t),(x,0))+\mathrm{Lip}(g_m)D((x,0),(y,s))=\\
&= \mathrm{Lip}(g_m)\big(D((x,t),(x,0))+D((x,0),(y,s))\big)\leq\\
&\leq \mathrm{Lip}(g_m)(|t|+d(x,y)+|s|)=\mathrm{Lip}(g_m)(d(x,y)+|t-s|)\leq\\
&\leq \sqrt{2}\cdot\mathrm{Lip}(g_m)\cdot D((x,t),(y,s))=\sqrt{2}\cdot\mathrm{Lip}(g_m)\cdot D(g_m^{-1}(p),g_m^{-1}(q))\leq\\
&\leq \sqrt{2}\cdot\mathrm{Lip}(g_m)\cdot\mathrm{Lip}(g_m^{-1})\cdot d(p,q)\leq\sqrt{2}\cdot\mathrm{biLip}(g_m)\cdot d(p,q)
\end{aligned}$$

and hence

$$D(G^{-1}(p),G^{-1}(q))\leq\sqrt{2}\cdot\max\big\{\mathrm{Lip}_b((G^+)^{-1})+\delta,\mathrm{Lip}_b((G^-)^{-1})+\delta\big\}\cdot S\cdot d(p,q).$$

It follows that

$$\text{the restriction of } G^{-1} \text{ to } \mathcal{B}_d(b, r_b) \text{ is Lipschitz,} \tag{50}$$

$$\mathrm{Lip}_b(G^{-1}) \leq \sqrt{2} \cdot \max\{\mathrm{Lip}_b((G^+)^{-1}) + \delta, \mathrm{Lip}_b((G^-)^{-1}) + \delta\} \cdot S$$

for each $\delta \in \mathbb{R}^+_*$ and hence

$$\mathrm{Lip}_b(G^{-1}) \leq \sqrt{2} \cdot \max\{\mathrm{Lip}_b((G^+)^{-1}), \mathrm{Lip}_b((G^-)^{-1})\} \cdot S.$$

In particular, we have that

$$\mathrm{Lip}^*(G^{-1}) \leq \sqrt{2} \cdot \max\{\mathrm{Lip}^*((G^+)^{-1}), \mathrm{Lip}^*((G^-)^{-1})\} \cdot S.$$

By (47) and (49), we obtain that

$$\mathrm{Lip}^*(G^{-1}) \leq (\sqrt{2})^{3k+2} \cdot (1 + L^{-1}) \cdot (3 + 2R)^k \cdot S \cdot \textstyle\prod_{n=1}^k \mathrm{biLip}^*(g_n)$$

as desired. The proof of Theorem 3L is complete.

Step III. Suppose now B is compact. Under this hypothesis, in *Step I*, we can use Theorem 1L* in place of Theorem 1L. In this way, we can assume that the homeomorphism G is also Lipschitz. From *Step II*, we know that G^{-1} is weakly Lipschitz. It remains to prove that G^{-1} is Lipschitz. By (50), for each $b \in B$, there exists $r_b \in \mathbb{R}^+_*$ such that $\mathcal{B}_d(b, r_b) \subset \Xi$ and $G^{-1}|_{\mathcal{B}_d(b,r_b)}$ is Lipschitz. Since B is compact, there exists a finite subset $\{b_1, \ldots, b_c\}$ of B such that $B \subset \bigcup_{i=1}^c \mathcal{B}_d(b_i, r_{b_i})$. Consider the finite open covering

$$\mathcal{C} := \{\Xi^+ \setminus B, \Xi^- \setminus B, \mathcal{B}_d(b_1, r_{b_1}), \ldots, \mathcal{B}_d(b_c, r_{b_c})\}$$

of the compact subspace Ξ of X. Since the restrictions of G^{-1} to each element of $\mathcal{C}$ are Lipschitz, point (2) of Lemma 2.3 ensures that G^{-1} is Lipschitz. This completes the proof of Theorem 3L*. □

Proof of Theorems 3L.** Let $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^{m-1}$ be the projection onto the first $m-1$ coordinates. For each $n \in \{1, \ldots, k\}$, denote by $\Xi_n : U_n \times (-1, 1) \longrightarrow V_n \times (-1, 1)$ the homeomorphism sending $(x, t) \in U_n \times (-1, 1)$ into $(\pi(\varphi_n(x)), t) \in V_n \times (-1, 1)$ and define the homeomorphism $g_n : U_n \times (-1, 1) \longrightarrow \Omega_n$ by setting $g_n := \varphi_n^{-1} \circ \Xi_n$. Proceeding as in the proof of Theorem 2L, one can show that $(\mathcal{U}, \{g_n\}_{n=1}^k)$ is a system of bi–Lipschitz local bi–collars of Y and it holds:

$$\mathrm{biLip}^*(g_n) \leq 2\big(\mathrm{biLip}^*(\varphi_n)\big)^2 \tag{51}$$

for each $n \in \{1, \ldots, k\}$. In particular, we have that

$$\max_{n\in\{1,\ldots,k\}} \mathrm{biLip}^*(g_n) \leq 2U^2. \tag{52}$$

By the General Separation Theorem (see [4, Theorem 27.10]), we know that $\mathbb{R}^m \setminus Y$ has exactly two connected components. This ensures that Y is two–sided and each system of bi–Lipschitz local bi–collars of Y, like $(\mathcal{U}, \{g_n\}_{n=1}^k)$, is two–sided coherent. In this way, we can apply Theorem 3L*, which, together with (51) and (52), implies immediately Theorem 3L**. □

Proof of Theorems 4L and 4L*. Let $\Pi : B \times (-1, 1) \longrightarrow B$ be the natural projection onto the first factor. Thanks to Theorem 3L, if the hypothesis of Theorem 4L are verified, then there exists a weakly bi–Lipschitz bi–collar $G : B \times (-1, 1) \longrightarrow \Omega$ of B. Define the map $\rho : \Omega \longrightarrow B$ by setting $\rho := \Pi \circ G^{-1}$. It is immediate to verify that Π is a Lipschitz map with $\mathrm{Lip}(\Pi) = 1$. It follows that ρ is a weakly Lipschitz tubular neighborhood of B with $\mathrm{Lip}^*(\rho) \leq \mathrm{Lip}^*(G^{-1})$ as desired. This proves Theorem 4L. Using Theorem 3L* instead of Theorem 3L, we obtain Theorem 4L*. □

References

[1] BROWN, M.: Locally flat imbeddings of topological manifolds. - Ann. of Math. (2) 75, 1962, 331–341.

[2] CONNELLY, R.: A new proof of Brown's collaring theorem. - Proc. Amer. Math. Soc. 27, 1971, 180–182.

[3] GHILONI, R.: The Hodge decomposition theorem for general three-dimensional vector fields, without cuts. - Technical Report UTM 731, Matematica, Università degli Studi di Trento, 2010, available at http://eprints.biblio.unitn.it/archive/00001873/

[4] GREENBERG, M. J.: Lectures on algebraic topology. - W. A. Benjamin, Inc., New York–Amsterdam, 1967.

[5] LUUKKAINEN, J., and J. VÄISÄLÄ: Elements of Lipschitz topology. - Ann. Acad. Sci. Fenn. Ser. A I Math. 3:1, 1977, 85–122.