A STRICT POSITIVSTELLENSATZ FOR DEFINABLE QUASIANALYTIC RINGS

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Abstract. Consider an expansion of the real field in which every unary definable continuous function can be ultimately majorized by a definable analytic function. We prove the strict Positivstellensatz for analytic functions which are definable in such structures. The methods also work for a large class of quasianalytic subrings of those rings of smooth functions which are definable in a polynomially bounded structure.

1. Introduction

The classical Positivstellensatz of Stengle, cf. [12], characterizes the positive polynomials on the basic closed semialgebraic set $F := \{f_1 \geq 0, \dots, f_k \geq 0\}$ as follows: A polynomial g is positive on F if and only if there are polynomials p_1, p_2 in the positive cone generated by the f_i and the sums of squares such that $p_1 g = p_2 + g^{2m}$.

Schmüdgen proved in [10] that if F is compact and g strictly positive on F, then p_1 is not needed, and Putinar verified additional conditions on the f_i for g belonging to the quadratic module generated by the f_i. In [1], Acquistapace, Andradas and Broglia proved the strict Positivstellensatz for (global) analytic functions on Euclidean spaces. In the present note, we prove a definable version of the strict Positivstellensatz of [1].

Let $\mathfrak{R}$ denote an expansion of the real field (see [4, 5, 6] for introductions to structures over the real field). By *definable*, we always mean *definable in* $\mathfrak{R}$ *with parameters from* $\mathbb{R}$, and functions are *definable* if their graphs are definable.

We consider structures $\mathfrak{R}$ which satisfy the following condition:

(A) For every definable continuous function $\phi : [0, \infty) \to \mathbb{R}$ there is a definable analytic function $\Phi : [-1, \infty) \to \mathbb{R}$ such that $\Phi \geq \phi$ on $[0, \infty)$.

This property is not very restrictive. No structure failing this property is known to the author; in particular, every known o-minimal expansion of the real field satisfies this property.

For a finite set of functions $f_1, \dots, f_k : \mathbb{R}^n \to \mathbb{R}$, we let

$$F := \cap_{i=1}^k \{f_i \geq 0\}.$$

We shall prove the following theorem.

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Theorem 1.1. *Assume that $\mathfrak{R}$ satisfies property (A). Let $g, f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ be definable and analytic, such that $g > 0$ on F and $F \neq \emptyset$. Then there are definable analytic functions $v_0, \ldots, v_k \in \mathcal{C}^\omega(\mathbb{R}^n, (0, \infty))$ such that*

$$g = v_0^2 + \sum_{i=1}^{k} v_i^2 f_i.$$

Denote by $\mathcal{C}^\infty_{def}(\mathbb{R}^n, \mathbb{R})$ the definable smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$. Suppose now that $\mathfrak{R}$ is additionally polynomially bounded; that is, every definable continuous function $\phi : [0, \infty) \to \mathbb{R}$ is bounded by some polynomial. Then $\mathcal{C}^\infty_{def}(\mathbb{R}^n, \mathbb{R})$ is *quasianalytic*; i.e., the Taylor homomorphism $T : \mathcal{C}^\infty_{def}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}[[X_1, \ldots, X_n]]$ mapping f to its Taylor series at 0 is injective, cf. [8]. In particular, every subring of $\mathcal{C}^\infty_{def}(\mathbb{R}^n, \mathbb{R})$ is quasianalytic.

Theorem 1.2. *Let $\mathfrak{R}$ be a polynomially bounded o-minimal expansion of the real field. Suppose $\mathcal{B}$ is a subring of $\mathcal{C}^\infty_{def}(\mathbb{R}^n, \mathbb{R})$, such that*

(a) $\mathbb{R}[X_1, \ldots, X_n] \subset \mathcal{B}$
(b) *if $f \in \mathcal{B}$, and $f > 0$, then $1/f$ and $\sqrt{f}$ belongs to $\mathcal{B}$.*

Let $g, f_1, \ldots, f_k \in \mathcal{B}$, such that $g_s > 0$ on F and $F \neq \emptyset$. Then there are strictly positive functions $v_0, \ldots, v_k \in \mathcal{B}$ such that

$$g = v_0^2 + \sum_{i=1}^{k} v_i^2 f_i.$$

The ring $\mathcal{C}^\omega_{def}(\mathbb{R}^n, \mathbb{R})$ is an example of such a ring $\mathcal{B}$.

Remark 1.3. If one considers differentiable or smooth functions when the exponential function is definable, then one obtains a far stronger Positivstellensatz, see [2, 7].

The proofs in [1] essentially use the fairly transcendental tool of analytic approximation of continuous functions with respect to the (strong) Whitney topology. So far, definable analytic approximation is only known for the semialgebraic structure, cf. [11]. By giving a rather explicit construction of the functions $v_0, \ldots, v_k$ of the above theorems we avoid approximation while preserving definability. Section 2 is devoted to the construction of these functions to prove both Theorem 1.1 and Theorem 1.2. In the final Section 3 we briefly discuss some generalizations and a consequence of our theorems.

2. Proofs

2.1. **Two lemmas.** We use $\|\cdot\|$ to denote the Euclidean norm, and $|\cdot|$ to denote the absolute value. We start by proving two elementary lemmas.

Lemma 2.1. *Let $\varepsilon : \mathbb{R}^n \to (0, \infty)$ be definable and continuous.*

(a) *If $\mathfrak{R}$ satisfies (A), then there is a definable $\mathcal{C}^\omega$ function $\rho : \mathbb{R}^n \to (0, 1)$ such that $\rho < \varepsilon$ on $\mathbb{R}^n$.*
(b) *If $\mathfrak{R}$ is polynomially bounded, then there is a strictly positive function $\rho \in \mathcal{B}$ such that $\rho < \varepsilon$ on $\mathbb{R}^n$.*

Proof. For $t \geq 0$ let

$$L(t) := \sup\left\{\frac{1}{\varepsilon(x)}; \|x\| \leq \sqrt{t}\right\}.$$

To prove (a), we notice that by (A) there is a definable function $\Phi : [-1, \infty) \to \mathbb{R}$ with $\Phi(t) > L(t)$. Set $\rho = 1/(1 + \Phi(\|x\|^2))$.

To prove (b), we observe that the polynomial boundedness of $\mathfrak{R}$ implies that there is a polynomial p such that $p(t) > L(t)$. Set $\rho := 1/(1 + p(\|x\|^2))$. □

Lemma 2.2. *Let $U \subset \mathbb{R}^n$ be a definable open set, and let $B \subset U$ be a definable closed set. Let $\varepsilon : U \to (0, \infty)$ be a definable continuous function. Then there exists a definable continuous function $\tilde{\varepsilon} : \mathbb{R}^n \to (0, \infty)$ such that $\tilde{\varepsilon} = \varepsilon$ on B.*

Proof. Let $\phi_1(x) = \operatorname{dist}(x, A)$, and let $\phi_2(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus U)$. Then

$$\tilde{\varepsilon} := \frac{\phi_1 + \varepsilon\phi_2}{\phi_1 + \phi_2}$$

satisfies the required properties. □

2.2. **Reduction step.** The proofs of Theorem 1.1 and Theorem 1.2 are similar. So if $\mathfrak{R}$ is an expansion of $\mathbb{R}$ which is not polynomially bounded but satisfies (A), we let $\mathcal{A} := \mathcal{C}^{\omega}_{def}(\mathbb{R}^n, \mathbb{R})$, and if $\mathfrak{R}$ is a polynomially bounded expansion of the real field, we let $\mathcal{A}$ be a subring $\mathcal{B}$ of $\mathcal{C}^{\infty}_{def}(\mathbb{R}^n, \mathbb{R})$ satisfying the conditions (a) and (b) of Theorem 1.2.

The following lemma is used to reduce the proof of the theorems to the case $k = 1$.

Lemma 2.3. *Let $g, f_1, \dots, f_k \in \mathcal{A}$ and let $g > 0$ on F, and $F \neq \emptyset$. Then there exist strictly positive function $s_1, \dots, s_k \in \mathcal{A}$ such that $h := \sum_{i=1}^{k} s_i^2 f_i$ satisfies $F \subset \{h \geq 0\} \subset \{g > 0\}$.*

Proof. Consider $\varepsilon : \mathbb{R}^n \setminus F \to \mathbb{R}$ defined by

$$\varepsilon := \frac{\sum_{i=1}^{k} \max(-f_i, 0)^2}{\sum_{i=1}^{k} |f_i|}.$$

This function is definable, continuous and strictly positive (particularly on $\{g \leq 0\}$). By Lemma 2.2, there is a definable continuous function $\tilde{\varepsilon} : \mathbb{R}^n \to (0, \infty)$ such that $\tilde{\varepsilon} = \varepsilon$ on $\{g \leq 0\}$. By Lemma 2.1, there is a $\rho \in \mathcal{A}$ such that $0 < \rho < \tilde{\varepsilon}$.

For $i = 1, \dots, k$, set

$$\phi_i := \frac{1}{2}\left(\sqrt{\rho^2 + f_i^2} - f_i\right).$$

Then we have

$$0 < \phi_i < \frac{\rho}{2} \text{ on } \{f_i \geq 0\},$$

and

$$|f_i| < \phi_i < |f_i| + \frac{\rho}{2} \text{ on } \{f_i \leq 0\}.$$

We notice that the functions ϕ_i are strictly positive. Hence, there are strictly positive functions $s_i \in \mathcal{A}$ such that $s_i^2 = \phi_i$. Define

$$h := \sum_{i=1}^{k} s_i^2 f_i.$$

Then $F \subset \{h \geq 0\}$. Let δ_i be the characteristic function of the set $\{f_i > 0\}$. Then we have on $\{g \leq 0\}$ that

$$\begin{aligned} h &= \sum_{i=1}^{k} \phi_i f_i \\ &\leq \sum_{i=1}^{k} (\max(-f_i, 0) + \delta_i \rho/2) f_i \\ &\leq \sum_{i=1}^{k} \max(-f_i, 0) f_i + \frac{\rho}{2} \sum_{i=1}^{k} |f_i| \\ &\leq -\sum_{i=1}^{k} \max(-f_i, 0)^2 + \frac{1}{2} \sum_{i=1}^{k} \max(-f_i, 0)^2 \\ &= \frac{-1}{2} \sum_{i=1}^{k} \max(-f_i, 0)^2 \\ &< 0 \end{aligned}$$

Hence $\{h \geq 0\} \subset \{g > 0\}$. □

2.3. **Proof of Theorem 1.1 and Theorem 1.2.** We are now able to prove the theorems.

Proof. By Lemma 2.3 there are strictly positive functions $s_1, \ldots, s_k \in \mathcal{A}$ such that $h = \sum_{i=1}^{k} s_i^2 f_i$ satisfies $F \subset \{h \geq 0\} \subset \{g > 0\}$. Hence $\{h \geq 0\} \cap \{g \leq 0\} = \emptyset$. By Lemma 2.1 and Lemma 2.2 we can select a strictly positive function $\rho \in \mathcal{A}$ such that

$$\begin{aligned} \rho &\leq \min\left(1, g+h, \frac{g}{\sqrt{h}}, \frac{g^2}{2h}\right) && \text{on } \{h \geq 0\}, \\ \rho &\leq \min\left(1, -h, \frac{h^2}{2\,|g|}\right) && \text{on } \{g \leq 0\}. \end{aligned}$$

Define the functions $\sigma_1, \sigma_2, \tau_1$ and τ_2 by

$$\begin{aligned} \tau_1 &:= \sqrt{\rho^2 + (g+h)^2}, \\ \tau_2 &:= \sqrt{\rho^2 + g^2}, \\ \sigma_1 &:= \frac{1}{2}\left(\sqrt{\rho^4 + g^2} + g\right), \\ \sigma_2 &:= \frac{1}{2}\left(\sqrt{\rho^4 + (g+h)^2} - (g+h)\right). \end{aligned}$$

Set

$$w := \frac{\sigma_1}{\tau_1} + \frac{\sigma_2}{\tau_2}.$$

This function is strictly positive and belongs to $\mathcal{A}$.

We claim that the function u given by

$$u := g - wh$$

is strictly positive. This will be proved in several steps.
step 1: Observe that on $\{g \le 0\}$ we have

$$\frac{\sigma_1}{\tau_1} \le \frac{\frac{1}{2}\rho^2}{\sqrt{\rho^2+(g+h)^2}} \le \frac{\rho}{2},$$

and on $\{h \ge 0\}$ we have

$$\frac{\sigma_2}{\tau_2} \le \frac{\frac{1}{2}\rho^2}{\sqrt{\rho^2+h^2}} \le \frac{\rho}{2}.$$

step 2 (a): On $\{h \ge 0\}$ we have

$$\begin{aligned} g\tau_1 - h\sigma_1 &= g\sqrt{\rho^2+(g+h)^2} - \frac{h}{2}\left(\sqrt{g^2+\rho^4}+g\right) \\ &> g\sqrt{\rho^2+(g+h)^2} - h\left(g+\frac{\rho^2}{2}\right) \\ &> g(g+h) - hg - \frac{1}{2}h\min\left(1,\frac{g^2}{h}\right) \\ &\ge \frac{g^2}{2}. \end{aligned}$$

Hence, the function $\varepsilon_1 : \{h \le 0\} \to \mathbb{R}$ defined by

$$\varepsilon_1 := \frac{g\tau_1 - h\sigma_1}{\tau_1} > \frac{g^2}{2\sqrt{\rho^2+(h+g)^2}} \ge \frac{g}{2\sqrt{2}}$$

is strictly positive.
step 2 (b): On $\{g \le 0\}$ we have

$$\begin{aligned} g\tau_2 - h\sigma_2 &= g\sqrt{\rho^2+h^2} - h\frac{1}{2}\left(\sqrt{(g+h)^2+\rho^4}-(g+h)\right) \\ &> g\sqrt{\rho^2+h^2} - h(-(g+h)) \\ &> -gh + hg + h^2 \\ &= h^2. \end{aligned}$$

Hence, the function $\varepsilon_2 : \{h \le 0\} \to \mathbb{R}$ defined by

$$\varepsilon_2 := \frac{\tau_2 g - \sigma_2 h}{\tau_2} > \frac{h^2}{\sqrt{g^2+h^2}} \ge -h$$

is strictly positive.
step 3: Verifying $u > 0$.
On $\{h \ge 0\}$ we have

$$u = g - \frac{\sigma_1}{\tau_1}h - \frac{\sigma_2}{\tau_2}h \ge \varepsilon_1 - \frac{\rho}{2}h \ge \frac{g^2}{2} - \frac{1}{2}\min\left(1,\frac{g^2}{2h}\right)h \ge \frac{g^2}{4} > 0.$$

On $\{g \le 0\}$ we have

$$u = g - \frac{\sigma_1}{\tau_1}h - \frac{\sigma_2}{\tau_2}h \ge \varepsilon_2 - \frac{\rho}{2}h \ge -h > 0.$$

On $\{g > 0\} \cap \{h < 0\}$ we evidently have $u = g - wh > 0$.

Finishing the proof of the theorems, we set $v_0 = \sqrt{u}$, and $v_i = \sqrt{w}s_i$, and obtain the equality

$$g = u + wh = v_0^2 + \sqrt{w}^2 \sum_{i=1}^{k} s_i^2 f_i = v_0^2 + \sum_{i=1}^{k} v_i^2 f_i.$$

□

3. Remarks and consequence

Remark 3.1. Theorem 1.2 remains true if we replace $\mathbb{R}$ by any real closed field R, and $\mathfrak{R}$ by any polynomially bounded definably complete expansion of R, cf. [9].

Remark 3.2. The proofs do not make use of the quasianalycity of the considered rings. They actually work for more general rings, as long as a corresponding statement of Lemma 2.1 holds true. Lemma 2.2 works (in an appropriate formulation) for continuous functions on a Banach space.

Remark 3.3. Let $d \in \mathbb{N}$. Suppose one stipulates that the ring $\mathcal{A}$ additionally satisfies the following property: If $f \in \mathcal{A}$ and $f > 0$ then $f^{1/2d} \in \mathcal{A}$. Then one can write $g = v_0^{2d} + \sum_{i=1}^{k} v_i^{2d} f_i$ in the statement of Theorem 1.2. For Theorem 1.1, this property is evidently satisfied.

Let $\mathrm{Spec}_r(\mathcal{A})$ denote the real spectrum of $\mathcal{A}$, see [3, Chp. 7]. Then $\mathbb{R}^n$ embeds canonically into $\mathrm{Spec}_r(\mathcal{A})$ by mapping x to the prime cone $\tilde{x}$ consisting of all $f \in \mathcal{A}$ such that $f(x) \geq 0$.

The following corollary is a particular case of the *Artin-Lang property.*

Corollary 3.4. *Let $f_1, \ldots, f_k \in \mathcal{A}$. Then*

$$\tilde{S} = \{\alpha \in \mathrm{Spec}_r(\mathcal{A}); f_1(\alpha) \geq 0, \ldots, f_k(\alpha) \geq 0\}$$

is not empty if and only if $S = \mathbb{R}^n \cap \tilde{S}$ is not empty.

Proof. Evidently, if $\tilde{S} = \emptyset$ then $S = \emptyset$. Suppose now that

$$S = \tilde{S} \cap \mathbb{R}^n = \{x \in \mathbb{R}^n; f_1(x) \geq 0, \ldots, f_k(x) \geq 0\} = \emptyset.$$

Then $-f_k$ is strictly positive on the set

$$F = \{x \in \mathbb{R}^n; f_1, (x) \geq 0, \ldots, f_{k-1}(x) \geq 0\}.$$

By Theorem 1.2 and Theorem 1.1, there are strictly positive functions $v_0, \ldots, v_{k-1} \in \mathcal{A}$ such that

$$-f_k = v_0^2 + \sum_{i=1}^{k-1} v_i^2 f_i.$$

Set $t_i = v_i/v_0$ for $i = 1, \ldots k$ and $t_k = 1/v_0$. Then

$$0 = 1 + \sum_{i=1}^{k} t_i^2 f_i.$$

Assume, for a contradiction, that there exists a $\beta \in \tilde{S}$. Then $f_i(\beta) \geq 0$ and $t_i^2(\beta) \geq 0$ for all i. Hence, we obtain the contradiction

$$0 = 1 + \sum_{i=1}^{k} t_i^2(\beta) f_i(\beta) \geq 1.$$

So $\tilde{S}$ is empty. □

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