CLOSURE THEOREM FOR PARTIALLY SEMIALGEBRAICS

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Abstract. In [3] was proven that the closure of a partially semialgebraic set is partially semialgebraic. The essential tool used in that proof was the regular separation property. Here we give another proof without using this tool, based on the semianalytic L-cone theorem (Theorem 5.2): a semianalytic analog of Cartan-Remmert-Stein lemma with parameters.

The authors began this joint paper in September 2002, but a heart attack cuts down the life of Professor Stanisław Lojasiewicz in November 2002. The second author wishes to honour Professor Stanisław Łojasiewicz with this paper.

1. Let M be an analytic manifold. When X is a finite dimensional real vector space of dimension n, then an analytic function $f : M \times X \to \mathbb{R}$ is said to be *X-polynomial on $M \times X$*, if for an open neighborhood U of any point of M, in some (then in any) linear coordinate system $X \to \mathbb{R}^n_t$ the restriction $f_{U\times X}$ is a polynomial in t with analytic coefficients on U.

A subset $E \subset M \times X$ is said to be *X-semialgebraic* if every point $a \in M$ has an open neighborhood U such that E_U is described in $U \times X$ by X- polynomials. This definition implies that the finite union, the finite intersection of X-semialgebraic sets is X-semialgebraic.

When Y is another finite dimensional real vector space, an analytic mapping $M \times X \to Y$ is said to be *X-polynomial mapping*, if in some (then in any) linear coordinate system in Y its components are X-polynomials.

If N is a real analytic manifold, Y a finite dimensional real vector space, $f : M \to N$ is an analytic mapping and $g : M \times X \to Y$ a X-polynomial mapping, then the inverse image by the mapping $(x, v) \to (f(x), g(x, v))$ of a Y-algebraic subset of $N \times Y$ is an X-semialgebraic subset of $M \times X$.

2. Let X be a topological space and let $\mathcal{A}$ be an algebra of real continuous functions on it. We say that a subset E of X is described by $\mathcal{A}$ if

$$E = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r} E_{ij}$$

where E_{ij} is of the form $\{f_{ij} > 0\}$ or $\{f_{ij} < 0\}$ or $\{f_{ij} = 0\}$ with $f_{ij} \in \mathcal{A}$.

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Let $S(\mathcal{A})$ be the class of subsets of X which are described by $\mathcal{A}$. Then $\mathcal{A}[t]$ is an algebra of real continuous functions on $X \times \mathbb{R}$. Let $\pi : X \times \mathbb{R} \to X$ be the natural projection.

The following facts has been proved in [2], p. 105-110.

Proposition 1. *Connected component theorem for $S(\mathcal{A})$ implies:*

(1) *Connected component theorem for $S(\mathcal{A}[t])$.*
(2) $E \in S(\mathcal{A}[t]) \Longrightarrow \pi(E) \in S(\mathcal{A})$.

If we take for $\mathcal{A}_U$ the ring of analytic functions on a relatively compact neighborhood U of a given point $a \in M$ which have analytic extensions on neighborhoods of $\overline{U}$, then, by the connected component theorem, each set of $S(\mathcal{A}_U)$ has a finite number of connected components. Hence, by induction, we obtain the following statement:

(1) Each set of $S(\mathcal{A}_U[t_1, \dots, t_m])$ has a finite number of component
(2) If $E \in S(\mathcal{A}_U[t_1, \dots, t_m])$, then $\pi(E)$ is semianalytic in M, where $\pi : M \times \mathbb{R}^k \to M$ is the natural projection for $k = 1, \dots, m$.

We obtain as consequences, using linear coordinate systems, the following connected components theorem and the following generalization of the Tarski-Seidenberg theorem.

Let X be a finite dimensional real vector space. We say the family $\mathcal{F}$ of subsets of $M \times X$ is *locally X-finite*, if every point of M has a neighborhood U such that the family of non-empty traces of the sets of $\mathcal{F}$ on $U \times X$ is finite. Observe that the union of any locally X-finite family of X-semialgebraic subsets of $M \times X$ is X-semialgebraic.

Connected components theorem. *All connected components of an X-semialgebraic set $E \subset M \times X$ are X-semialgebraic and their family is locally X-finite. Any open-closed subset of E is X-semialgebraic*

Now, let Y be a finite dimensional real vector space and let $\pi : M \times X \times Y \to M \times X$ be the natural projection.

Tarski-Seidenberg Theorem. *If $E \subset M \times X \times Y$ is $(X \times Y)$-semialgebraic, then $\pi(E)$ is X-semialgebraic (in $M \times X$).*

3. Let M be a real analytic manifold and let L be a finite dimensional real vector space. We say that a semianalytic $E \subset M \times L$ is a *L-cone* if each fiber E_x is a cone, for $x \in M$. This is equivalent to say that $c_\lambda(E) = E$ for $\lambda > 0$, where $c_\lambda = id_M \times \lambda id_L : M \times L \to M \times L$.

We have the following straightforward properties:

(1) The closure of an L-cone is an L-cone.
(2) The finite union, intersection and difference of L-cones are L-cones.
(3) The subset of points of dimension k of a semianalytic L-cone is a semianalytic L-cone.
(4) The set of smooth points of dimension k of an L-cone is an semiananlytic L-cone.

We call *L-conical germ* any set germ A at $(a, 0) \in M \times L$, if $c_\lambda(A) = A$ for any $\lambda \in \mathbb{R}$, $\lambda > 0$. Clearly, the germ at $(a, 0)$, with $a \in M$, of any L-cone C in $M \times L$ is

L-conical, and we have $\dim_{(a,0)} C = \dim(C)_{(a,0)}$. Observe that for L-cones $C,\ D$, we have

$$C \subset D \Longleftrightarrow \text{for all } a \in M\ ,\ (C)_{(a,0)} \subset (D)_{(a,0)}$$

and also

$$C = D \Longleftrightarrow \text{for all } a \in M\ ,\ (C)_{(a,0)} = (D)_{(a,0)}\ .$$

Lema 1. *For any L-conical semianalytic germ A at $(a,0) \in M \times L$ there is a semianalytic L-cone C in $U \times L$, where U is an open neighbordhood of a such that $(C)_{(a,0)} = A$. If A is analytic, so is C.*

Proof. Let E be a semianalytic representative of A in a neighborhood $U \times \Omega$ of $(a,0)$. For any $u \in U$ and $v \in L$ with $|v| = 1$ there is a maximal segment $(0, r(u,v))v$ which is contained in E_u or disjoint with E_u ([1]). As A is L-conical, we must have $r = \inf\{r(u,v) : u \in W, |v| = 1\} > 0$, with a relatively compact semianalytic neighborhood $W \subset U$ of a. Then, if $0 < s < r$, the set

$$D = E \cap (W \times \{|x| < s\}) \subset M \times L$$

is a union of some segments $u \times (0,s)v$ or $u \times [0,s)v$, with $|v| = 1$. It follows that $C = \bigcup_{\lambda > 0} c_\lambda(D)$ is a semianalytic L-cone ([2]) such that $C_{(a,0)} = A$.

We will need the following fact:

Proposition 2. *Any decreasing sequence E_ν of analytic subsets of M is locally stationary, i.e. each point $a \in M$ has a neighborhood U such that the sequence $E_\nu \cap U$ is stationary.*

Proof. Since at any point $c \in M$ the ring of analytic germs at c, $\mathcal{A}_c$, is noetherian, the sequence of germs $(E_\nu)_0$ is stationary i. e. for some p we have $(E_\nu)_0 = (E_p)_0$ for $\nu \geq p$. Taking a stratification at c, compatible with E_p ([3]), i. e. such that $E_p \cap U$ is the union of some of its leaves, and using a sort of the identity principle: *if an open non empty subset of a connected analytic leaf Λ is contained in E_p, then* $\Lambda \subset E_p$, we get the result.

We have the following real analog of Cartan-Remmert-Stein lemma with parameters.

We say that a subset E of $M \times L$ is *L-algebraic* if each $a \in M$ has an open neighborhood U such that E is defined (by equalities) by polynomials in $x \in L$ with analytic coefficients in U (always it can be defined by one function).

Theorem 1. *Each analytic L-cone in $M \times L$ is L-algebraic.*

Proof. Take any $a \in M$. Our L-cone E is defined in some open neighborhood $W = U \times V$ of $(a,0)$ by an analytic function $f = \sum_{i=0}^{\infty} h_i$, where h_i is a form of degree i with analytic coefficients in U. For any $(x,v) \in U \times V$ we have the equivalence:

$$f(x,v) = 0 \iff h_i(x,v) = 0\ ,\ i = 0,1,2,\dots\ .$$

In fact, if $f(x,v) = 0$, then, as E is an L-cone, we have $0 = f(x,tv) = \sum_{i=0}^{\infty} h_i(x,v)t^i$ for $t > 0$ small enough, hence $h_i(x,v) = 0$, $i = 0,1,2,\dots$. The

[1] Since $E_u \cap \mathbb{R}v$ is semiananlytic.

[2] Its germ at any point is the germ of some $c_\lambda(D)$ (at this point).

[3] E. g. a normal one (see [4]).

sequence of analytic sets $E_k = \{(x,v) \in U \times V : h_i(x,v) = 0, i = 1,\dots,k\}$ is decreasing, hence by the proposition 2, it must be locally stationary, so, after making smaller W, we have $\{f = 0\} \cap W = E_k$ for some k. Thus the sets E and E_k are equal in W, and so, being L-cones, they must be equal in $U \times L$.

Proposition 3. *For any semianalytic germ S at a point $a \in M$, there exists the smallest analytic germ at a which contains S. Moreover they have the same dimension.*

Proof. In fact, take the ideal $\mathcal{I}$ of the ring $\mathcal{A}$ of germs of analytic functins at a which vanish on S. As $\mathcal{A}$ is noetherian, the ideal $\mathcal{I}$ is generated by germs at a of the analytic functions $f_1,\dots,f_s$ in a neighborhood of a. Then the analytic germ:

$$\{f_1 = \cdots = f_s = 0\}_a$$

is the required one. The second part follows by taking a normal stratification compatible with a representant of S.

Proposition 4. *Any semianalytic L-cone is contained in an algebraic L-cone of the same dimension.*

Proof. Let E be our semianalytic L-cone. Let $a \in M$. Following the proposition 3 we take the smallest analytic germ S which contains $E_{(a,0)}$. It is L-conical since also $c_\lambda(S)$ is such a germ (in view of $c_\lambda(E_{(a,0)}) = E_{(a,0)}$. By lemma 1 we take the analytic L-cone C such that $C_{(a,0)} = S$. It has the same dimension as E and it is algebraic, by theorem 1.

4. We will need now some other facts from semianalytic geometry.

For any subset E of a topological space X, we define a decreasing sequence of closed sets $V_i = V_i(E)$, $i = 1,\dots,$ as follows. We define by recursion a sequence E_i, setting $E_0 = E$ and $E_{i+1} = \overline{E_i} \setminus E_i$, and we put $V_i = V_i(E) = \overline{E_i}$. In particular, $V_0 = \overline{E}$, and we have $V_0 \supset V_1 \supset \dots$.

We have the following

Lemma 2. *If $V_{2r} = \emptyset$, then $E = (V_0 \setminus V_1) \cup \cdots \cup (V_{2r-2} \setminus V_{2r-1})$.*

In fact $E_i = \overline{E_i} \ \setminus \ E_{i+1}$, and also

$$(V_i \setminus V_{i+1}) \cup (V_{i+1} \setminus E_{i+1}) = \overline{E_i} \ \setminus \ \overline{E_{i+1}} \ .$$

Hence, for p even, we have:

$$E_{2r-p} = (V_{2r-p} \ \setminus \ V_{2r-p+1}) \cup (V_{2r-p+2} \ \setminus \ V_{2r-p+3}) \cup \cdots \cup (V_{2r-2} \ \setminus V_{2r-1}) \ ,$$

and we get the result for $p = 2r$.

Proposition 5. *A set $E \subset M$ is semianalytic if and only if the sets $V_i = V_i(E)$ are closed semianalytic and $V_s = \emptyset$ for some s. Then V_{i+1} is nowhere dense in V_i, $i = 0,1,\dots,$ so $\dim V_i \le \dim E$, and we have $V_i = \emptyset$ for $i > n = \dim M$. If $2r > n$, then*

$$E = (V_0 \setminus V_1) \cup \cdots \cup (V_{2r-2} \setminus V_{2r-1}) \ . \qquad (*)$$

Observe that if E is a cone, so are the V_i's.

Proof. Indeed, by lemma 2, the condition is sufficient. Now suppose that the set E is semianalytic. Then the sets V_i are semianalytic, V_{i+1} is nowhere dense in V_i, $i = 0, 1, \dots$ and $V_i = \emptyset$ for $i > n$ (see [4], II, 5 and 7). Applying the lemma 2 we get the formula $(*)$. The last statement follows by the definition of $V(E)$.

Proposition 6. *If the set E is semianalytic in M, then the set*

$$\{x \in M : \dim_x E = k\}$$

is semianalytic

In fact, it is sufficient to take a distinguished (or normal) stratification of an interval Q, compatible with E (see [1]). Then the points of our set in Q are precisely those which belong to closures (in Q) of leaves of dimension k and do not belong to the closure of any leaf of dimension $> k$.

Remark 1. *If a semianalytic set is of constant dimension k, the same holds for its closure.*

5. We have the following semianalytic analog of Cartan-Remmert-Stein lemma with parameters.

Theorem 2. *Any semianalytic L-cone is L-semialgebraic.*

Proof. Let E be a semianalytic L-cone in $M \times L$. We proceed by induction on the dimension k of E. By proposition 5, we have

$$E = (F_0 \setminus F_1) \cup \cdots \cup (F_{p-1} \setminus F_p) \ ,$$

where F_i are closed semianalytic L-cones of dimension $\leq k$. In view of the proposition 6 and the remark 1, for each $F = F_i$ we have

$$F = S_k \cup \cdots \cup S_0 \ ,$$

with S_i closed semianalytic L-cone of constant dimension i.

By proposition 4, any L-cone S_i is contained in an L-algebraic L-cone C_i of dimension i. Let S_i^* the set of non smooth points of S_i. It is a closed semianalytic L-cone of dimension $< i < k$, and it is L-semialgebraic by the induction hypothesis. Then $S_i \setminus S_i^*$ is an analytic submanifold of dimension i; it is dense in the set S_i (as the last one is of constant dimension i). Next the set C_i^* of points of C_i which are not smooth ones of dimension i, is a closed L-semialgebraic L-cone of dimension $< i$, and $C_i \setminus C_i^*$ is an analytic submanifold.

The set $T = S_i^* \cup C_i^*$ is closed semialgebraic of dimension $< i$. Then the set $S_i \setminus T$ is closed in the set $C_i \setminus T$, but it is also open (in the last one) as both are analytic submanifolds of dimension i. Hence the set $S_i \setminus T$ is a locally L-finite union of connected components of $C_i \setminus T$, and then it is L-semalgebraic. So, since the set S_i is the closure of $S_i \setminus T$, it must be L-semialgebraic. Consequently, so are the F_i's and E.

6. Let X, Y be finite dimensional real vector spaces. Consider the hyperplane $H = X \times Y \times 1 \subset X \times Y \times \mathbb{R}$. Let F be a subset of $X \times Y$, and put

$$c(F) = \bigcup_{(x,y)\in F} \{x\} \times \mathbb{R}_+(y,1) ,$$

with $\mathbb{R}_+ = (0, \infty)$. Hence, $c(F) \cap H = F \times 1$.

Remark 2. Let $E \subset X \times Y$ a Y-cone, then

$$\tilde{E} = \{(x, y, z, t) \mid (x, y) \in E, \; z = ty, \; t > 0\} \subset X \times Y \times Y \times \mathbb{R}$$

is a $(Y \times \mathbb{R})$-cone, and $c(E) = \mu(\tilde{E})$, where $\mu(x, y, z, t) = (x, z, t)$, for (x, y, z, t) in $X \times Y \times Y \times \mathbb{R}$. Thus, by Tarski-Seidenberg theorem, if E is a Y-cone Y-semialgebraic then $c(E)$ is a $(Y \times \mathbb{R})$-cone Y-semmialgebraic.

Remark 3. Let be $M = X \times Y \times \mathbb{R}_+$ and consider $E \subset X \times Y$. Then, $Adh_M(c(E)) = c(Adh_{X\times Y}E)$.

Thus, it follows the equivalence:

Proposition 7. *Let be $E \subset X \times Y$ any Y-cone. Then:*
E is Y-semialgebraic if and only if $c(E)$ is semianalytic.

Proof. Consider the $(Y \times \mathbb{R})$-cone $\tilde{E}$ as in the remark 2. It is $(Y \times Y)$-semialgebraic. By Tarski-Seidenberg theorem $\mu(\tilde{E})$ is Y-semialgebraic, hence $c(E)$ is semianalytic. Conversely, since $c(E) \cap H = E \times 1$, the set E is a semianalytic set. Hence, by theorem 2 in **5**, E is Y-semialgebraic.

Closure theorem. *The closure of any Y-semialgebraic subset of $X \times Y$ is Y-semialgebraic.*

Proof. Take a Y-semialgebraic set A. Put $E = c(A)$. The set E is a Y-cone and it is Y-semialgebraic. By proposition 7, the cone $c(E)$ is semianalytic. Thus $\overline{c(E)}$ is semianalytic (by the closure theorem for semianalytic sets, see [4], II.5, cor. 5.2). The set $\overline{E}$ is a Y-cone, and it is also Y-semialgebraic, because of the remark 3 and the proposition 7. The remark 3 implies that $c(\overline{A})$ is Y-semialgebraic. So it is $\overline{A}$, since $\overline{A} \equiv c(\overline{A}) \cap H$.

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