A proof of the valuation property and preparation theorem

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Abstract

The purpose of this article is to present a short model-theoretic proof of the valuation property for a polynomially bounded, o-minimal theory T. The valuation property was conjectured by van den Dries [1], and proved for the polynomially bounded case by van den Dries–Speissegger [4] and for the power bounded case by Tyne [11].

Our proof uses the transfer principle for the theory T_{conv} (theory T with an extra unary symbol denoting a proper convex subring) which — together with quantifier elimination — is due to van den Dries–Lewenberg [2]. The main tools applied here are saturation, the Marker–Steinhorn theorem on parameter reduction [8] and heir-coheir amalgams (see e.g. [6], Chap. 6).

The significance of the valuation property lies to a great extent in its geometric content: it is equivalent to the preparation theorem (which says, roughly speaking, that every definable function of several variables depends piecewise on any fixed variable in a certain simple fashion). This theorem originates in Parusiński [9, 10] for subanalytic functions, and in Lion–Rolin [7] for logarithmic-exponential functions. Van den Dries–Speissegger [5] have proved the preparation theorem in the o-minimal setting (for functions definable in a polynomially bounded structure or logarithmic-exponential over such a structure). Also, the valuation property makes it possible to establish quantifier elimination for polynomially bounded expansions of the real field $\mathbb{R}$ with exponential function and logarithm (see [4, 3]).

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1. Preliminaries. Throughout this article we deal with a polynomially bounded o-minimal theory T in a first-order language $\mathcal{L}$ with field of exponents K (being a subfield of the field $\mathbb{R}$ of reals). The word "definable" in a structure $\mathcal{R}$ always means "definable with parameters from $\mathcal{R}$"; "definable with no parameters" is called "0-definable". It is well known that one can always extend by definitions $\mathcal{L}$ and T to:

$\mathcal{L}^{df} := \mathcal{L}$ augmented by a new function symbol $f_\varphi(\bar{x})$ for each $\mathcal{L}$-formula $\varphi(\bar{x}, y)$ such that $T \vdash \forall\, \bar{x}\, \exists!\, y \ \ \varphi(\bar{x}, y)$;

$T^{df} := T$ extended by the new defining axioms $\varphi(\bar{x}, f_\varphi(\bar{x}))$.

Every model $\mathcal{R}$ of T expands uniquely to a model $\mathcal{R}^{df}$ of T^{df}. Since the theory T has definable cell decomposition, T^{df} has quantifier elimination, and since T has definable Skolem functions (which follows from cell decomposition as well), T^{df} has universal axiomatization. In consequence, T has a prime model $\mathcal{P}$ which has a unique elementary embedding into every model $\mathcal{R}$ of T; the image of $\mathcal{P}$ in $\mathcal{R}$ consists of the interpretations of all constant symbols of the language $\mathcal{L}^{df}$.

If $\mathcal{R}$ is an elementary substructure of a model $\mathcal{S}$ of T: $\mathcal{R} \prec \mathcal{S}$, then $\mathcal{R}^{df}$ is a substructure of $\mathcal{S}^{df}$ in the language $\mathcal{L}^{df}$: $\mathcal{R}^{df} \subset \mathcal{S}^{df}$. For a subset $A \subset \mathcal{S}$, $\mathcal{R}\langle A \rangle$ denotes the definable closure of A over $\mathcal{R}$ in $\mathcal{S}$, i.e. the substructure of $\mathcal{S}^{df}$ generated by $\mathcal{R} \cup A$ in the language $\mathcal{L}^{df}$; $\mathcal{R}\langle A \rangle$ is, of course, an elementary substructure of $\mathcal{S}$. The operation of definable closure fulfils the ordinary axioms for span operation (in particular the Steinitz exchange property), whence one can define in an ordinary fashion rank, $\mathrm{rk}\,(\mathcal{R})$, or relative rank, $\mathrm{rk}\,(\mathcal{S}/\mathcal{R})$.

Consider two ordered fields $\mathcal{R} \subset \mathcal{S}$. We say that $\mathcal{R}$ is Dedekind complete or tame in $\mathcal{S}$ if one of the three equivalent conditions is satisfied:

i)the trace on $\mathcal{R}$ of every interval in $\mathcal{S}$ is an interval in $\mathcal{R}$;

ii) the cut made in $\mathcal{R}$ by every element $s \in \mathcal{S}$ is rational;

iii) for each $\mathcal{R}$-bounded element $s \in \mathcal{S}$, there is a unique element $r \in \mathcal{R}$ such that $s - r$ is an $\mathcal{R}$-infinitesimal; we call $r := \mathrm{st}\,(s)$ the standard part of the element s.

Marker–Steinhorn theorem on parameter reduction. *Consider o-minimal structures $\mathcal{R} \prec \mathcal{S}$ with $\mathcal{R}$ Dedekind complete in $\mathcal{S}$. Then the trace $X \cap \mathcal{R}^n$ of every definable set $X \subset \mathcal{S}^n$ in $\mathcal{S}^n$ is definable in $\mathcal{R}^n$. In other words, the trace $X \cap \mathcal{R}^n$ can be defined by means of parameters from $\mathcal{R}$ only.*

In this section we shall make use of the following well-known

Corollary. (cf. [8, 1]) *Under the previous assumptions, if* $f : \mathcal{S}^n \longrightarrow \mathcal{S}$ *is a definable function, then the sets*

$$E_{-\infty} := \{\bar{x} \in \mathcal{R}^n : f(\bar{x}) < \mathcal{R}\}, \quad E_{+\infty} := \{\bar{x} \in \mathcal{R}^n : f(\bar{x}) > \mathcal{R}\}$$

are definable, and the function

$$\mathrm{st}\,(f(\bar{x})) : \mathcal{R}^n \setminus (E_{-\infty} \cup E_{+\infty}) \longrightarrow \mathcal{R}$$

is definable.

Consider now the theory T_{conv} of pairs $(\mathcal{R}, V)$, where V is a convex subring of a model $\mathcal{R}$ of T, in the language $\mathcal{L}$ with an extra unary relation symbol to denote V (cf. [2]). Every maximal elementary substructure $\mathcal{R}'$ of $\mathcal{R}$ contained in V is Dedekind complete and cofinal in $\mathcal{R}$, and isomorphic to the residue field $\bar{V}$ of V; V is the convex hull of $\mathcal{R}'$ in $\mathcal{R}$. Van den Dries–Lewenberg [2] proved a relative version of quantifier elimination for T_{conv}:

If T has quantifier elimination and universal axiomatization, then T_{conv} has quantifier elimination.

It follows that T_{conv} is a complete theory, and thus we have at our disposal the transfer principle: in order to prove a theorem expressible in the first-order language $\mathcal{L}_{conv}$ for all models of T_{conv}, it suffices to prove it for one particular model.

The convex subring V is a valuation ring in $\mathcal{R}$ with maximal ideal $\mathfrak{m} = \mathfrak{m}_V$ and valuation group $\Gamma = \Gamma_V$; let v denote the induced valuation of the field $\mathcal{R}$. We now give a simple proof for the following proposition on stabilization of valuation due to van den Dries [1]. Our proof makes use of the transfer principle and piecewise uniform asymptotics only.

Proposition 1 (on stabilization of valuation). *If* $f : \mathcal{R} \longrightarrow \mathcal{R}$ *is a definable function, then there exists* $u \in V$ *such that for all* $x \in V$, $x \geq u$, *we have*

$$v(f(x)) = v(f(u)) \quad \textit{or equivalently} \quad \frac{f(x)}{f(u)} \in V \setminus \mathfrak{m}.$$

The function f is of the form

$$f(x) = g(x, r_1, \dots, r_m),$$

where g is a 0-definable function and $\bar{r} \in \mathcal{R}^m$ are parameters. So we shall prove the statement for all parameters $\bar{r}$ from the model under consideration, g being fixed. Since the assertion is expressible in the first-order language of the theory T_{conv}, we may assume that $\mathcal{R} = \mathcal{P}\langle a \rangle$, where $|\mathcal{P}| < a$ and $V = \widehat{\mathcal{P}}$ is the convex hull of $\mathcal{P}$ in $\mathcal{R}$. Then every parameter $r_i = h_i(a)$ for a 0-definable function $h_i : \mathcal{P} \longrightarrow \mathcal{P}$. Putting

$$k(x, y) := g(x, h_1(y), \dots, h_m(y)),$$

we are thus reduced to considering functions $f(x) = k(x, a)$, $k : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}$ being 0-definable.

In this case, in view of piecewise uniform asymptotics for polynomially bounded o-minimal theories, there exist an exponent $\mu \in K$, a $u \in \mathcal{P}$ and a 0-definable function $c(x)$ such that for every $x \in \mathcal{P}$, $x \geq u$, we have

$$\frac{k(x, y)}{c(x) y^{\mu}} \to 1 \ \text{ when } \ y \to \infty, \ y \in \mathcal{P}.$$

Hence for every $x, \epsilon \in \mathcal{P}$, $x \geq u$, $\epsilon > 0$, we have

$$\left| \frac{k(x, y)}{c(x) y^{\mu}} - 1 \right| < \epsilon \quad \text{when} \ \ y \in \mathcal{P}, \ y \gg 0.$$

It follows by overspill that

$$\left| \frac{k(x, a)}{c(x) a^{\mu}} - 1 \right| < \epsilon \quad \text{for every} \ \ x \in V, \ x \geq u.$$

Consequently, $v(f(x)) = v(k(x, a)) = \mu v(a)$ for all $x \in V$, $x \geq u$, as desired.

Let $f, g : \mathcal{R} \longrightarrow \mathcal{R}$ be definable functions, and $\pi : V \longrightarrow \bar{V}$ the canonical mapping onto the residue field $\bar{V}$ of V. We say that f and g are asymptotic on V, $f \underset{V}{\sim} g$, if $f(x) = g(x) = 0$ for $x \in V$, $x \gg 0$, or

$$\frac{f(x)}{g(x)} \in V \ \text{ for } \ x \in V, x \gg 0, \ \text{ and } \ \pi \left(\frac{f(x)}{g(x)} \right) \to 1 \ \text{ when } \ x \to \infty, x \in V.$$

Corollary. *There exist* $\lambda \in K$ *and* $c \in \mathcal{R}$ *such that* $f(x) \underset{V}{\sim} cx^{\lambda}$. *Consequently, for any* $\epsilon = 1/v$ *with* $v \in V$, $v > 0$, *there exist* $u, w \in \mathcal{R}$, $u \in V$, $w > V$, *such that* $\left|\frac{f(x)}{cx^{\lambda}} - 1\right| < \epsilon$ *for all* $x \in [u, w]$.

Indeed, the maximal elementary substructure $\mathcal{R}' \simeq \bar{V}$ of $\mathcal{R}$ contained in V is Dedekind complete in $\mathcal{R}$. Taking $u \in \mathcal{R}'$ as in Proposition 1, it follows from the corollary to the Marker–Steinhorn theorem that the function

$$\mathrm{st}\left(\frac{f(x)}{f(u)}\right) : \mathcal{R}' \longrightarrow \mathcal{R}'$$

is $\mathcal{R}'$-definable. Then it is asymptotic to $c'x^{\lambda}$ for some $\lambda \in K$ and $c' \in \mathcal{R}'$:

$$\mathrm{st}\left(\frac{f(x)}{f(u)c'x^{\lambda}}\right) \to 1 \quad \text{when} \quad x \to \infty,\, x \in \mathcal{R}'.$$

Putting $c := f(u)c'$ completes the proof.

Consider now a model $(\mathcal{R}, V)$ of the theory T_{conv}, the valuation v of which has a finite rank $d < \infty$. This means that the value group $\Gamma = \Gamma_V$ has $d+1$ isolated subgroups of the form:

$$\Gamma_0 = (0) \subset \Gamma_1 \subset \Gamma_1 \oplus \Gamma_2 \subset \ldots \subset \Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \ldots \oplus \Gamma_d,$$

where the subgroups Γ_i are archimedian such that $\Gamma_1^+ < \Gamma_2^+ < \ldots < \Gamma_d^+$. We have, of course, a one-to-one correspondence between these isolated subgroups and the prime ideals $\mathfrak{p}_i$ in V, as well as the convex subrings V_i such that $V \subset V_i \subset \mathcal{R}$:

$$\mathfrak{p}_i = \{x \in \mathcal{R} : v(x) > \Gamma_i\}, \quad V_i = \{x \in \mathcal{R} : v(x) > -\Gamma_{i+1}\} = V_{\mathfrak{p}_i};$$

$$\mathfrak{p}_0 = \mathfrak{m} \supset \mathfrak{p}_1 \supset \ldots \supset \mathfrak{p}_{d-1} \supset \mathfrak{p}_d = (0), \quad V_0 = V \subset V_1 \subset \ldots \subset V_{d-1} \subset V_d = \mathcal{R}.$$

The valuation group of V_i is isomorphic to $\Gamma_{i+1} \oplus \ldots \oplus \Gamma_d$; the valuation group of $V_d = \mathcal{R}$ is (0).

Observation. There exists an elementary extension $(\mathcal{R}^*, V^*)$ of $(\mathcal{R}, V)$, the value group Γ^* of which is of the form

$$\Gamma^* = \underbrace{\mathbb{R} \oplus \ldots \oplus \mathbb{R}}_{d \text{ times}}$$

Such an elementary extension can be obtained by a successive adjunction of elements from an $\aleph_0$-saturation of $(\mathcal{R}, V)$. Here we sketch that procedure.

Every archimedian group Γ_i may be regarded as a subgroup of the additive group of real numbers $\mathbb{R}$; we may assume that $1 \in \Gamma_i$. Take any number $\delta \in \mathbb{R} \setminus \Gamma_i$; δ makes an irrational cut C in Γ_i. One can lift the cut C to a unique cut $\widetilde{C} := \{x \in \mathcal{R} : x \leq 0 \text{ or } (x > 0, v(x) > C)\}$ in $\mathcal{R}$, and next adjoin to $\mathcal{R}$ an element a which realizes the cut $\widetilde{C}$. We get an elementary extension $(\mathcal{R}\langle a\rangle, W)$ of $(\mathcal{R}, V)$; clearly, $w(a)$ realizes the cut C in Γ_i too. One must show that the valuation w with value group Γ_W obtained in this fashion is also of rank d. This holds due to the control over definable functions in the vicinity of the cuts made by the convex subrings V_i in $\mathcal{R}$ — as described in the corollary to Proposition 1. Indeed, every definable function $f : \mathcal{R} \longrightarrow \mathcal{R}$ is asymptotic in each V_i to a function cx^λ. Hence $\left|\frac{f(x)}{cx^\lambda} - 1\right| < 1/2$ for all x from an interval $[u, w]$ with $u \in V_i$, $w > V_i$, and this inequality extends to the cuts made by the subrings V_i in $\mathcal{R}$. Therefore, for every $\mathcal{R}$-definable function g, the element $b = g(a)$ can realize none of those cuts, and consequently $w(b) = w(g(a))$ can realize no cut in Γ made by the isolated subgroups Γ_i. Otherwise the element $w(b)$ would generate yet another isolated subgroup of Γ_W. Further, if $a = f(b)$, we would get $\left|\frac{f(a)}{ca^\lambda} - 1\right| < 1/2$, and thus

$$w(a) = v(c) + \lambda w(b) \in \Gamma + Kw(b)$$

whence $w(b) \in \Gamma + Kw(a)$. But this is impossible since the group $\Gamma + Kw(a)$, on a par with the group Γ, has exactly $d + 1$ isolated subgroups whenever $w(a)$ defines an irrational cut in Γ.

Repeating successively the above procedure for each subgroup Γ_i and all real numbers, we obtain an increasing chain of elementary extensions (of cardinality $\leq$ power of the continuum). By the Tarski–Vaught lemma, the union of this chain is the desired elementary extension $(\mathcal{R}^*, V^*)$ of $(\mathcal{R}, V)$.

Applying the transfer principle and saturated models, we shall prove the following

Proposition 2. *If a definable function $f : \mathcal{R} \longrightarrow \mathcal{R}$ is constant on no interval in $\mathcal{R}$, then there exist $s \in \mathcal{R}$, $\lambda \in K \setminus \{0\}$ and $c \in \mathcal{R}$ such that*

$$f(x) - s \underset{V}{\sim} cx^\lambda.$$

Consequently, for any $\epsilon = 1/v$ with $v \in V$, $v > 0$, there exist $u, w \in \mathcal{R}$, $u \in V$, $w > V$, such that $\left|\frac{f(x)-s}{cx^\lambda} - 1\right| < \epsilon$ for all $x \in [u, w]$.

In view of the corollary to Proposition 1, our statement is equivalent to the following first-order sentence

$$\exists\, s \in \mathcal{R}\ \forall\, u \in V\ \exists\, x_1, x_2 \in V,\ x_1, x_2 > u \qquad \left| \frac{f(x_1) - s}{f(x_2) - s} \right| > 2;$$

here the number 2 may be replaced by any real number > 1. Via the transfer principle, it suffices to consider one model of the theory T_{conv}. Take a model $(\mathcal{R}, V)$ with $V = \widehat{\mathcal{P}} =$ convex hull of the prime model $\mathcal{P}$ in $\mathcal{R}$, where $\mathcal{R}$ is an α^+-saturated model of the theory T with $\alpha =$ cofinality of $\mathcal{P}$. For simplicity we confine ourselves to the case $\alpha = \aleph_0$; the general case runs the same way, but with transfinite induction instead of an ordinary induction argument.

We prove Proposition 2 for the above model by reductio ad absurdum. Suppose the contrary, i.e. for any $s \in \mathcal{R}$ the function $f(x) - s \underset{V}{\frown} c$ is asymptotic to a constant function c, i.e. $\lambda = 0$; obviously, $c \neq 0$.

We then assert that $f(x) - s \in c(1 + \mathfrak{m})$ for all $x \in V$, $x \gg 0$. For otherwise, if $\mathcal{R}' \simeq \bar{V}$ is a maximal elementary substructure of $\mathcal{R}$ contained in V, the function

$$\mathrm{st}\left(\frac{f(x) - s}{c} \right) : \mathcal{R}' \longrightarrow \mathcal{R}',$$

would be asymptotic to 1 but $\neq 1$ ultimately in $\mathcal{R}'$. Hence $\left(\frac{f(x)-s}{c} - 1 \right)$ would be asymptotic to dx^λ with $d \in \mathcal{R}'$, $\lambda \in K$, $\lambda < 0$, and consequently

$$f(x) - s - c \underset{V}{\frown} c\, d\, x^\lambda,$$

contrary to our hypothesis. Therefore $\mathrm{st}\left(\frac{f(x)-s}{c} \right) = 1$ ultimately in $\mathcal{R}'$, and thus $f(x) - s \in c(1 + \mathfrak{m})$ for all $x \in V$, $x \gg 0$, as asserted.

Take any $a_0 \in V$; then there is some $c_0 \in \mathcal{R}$ such that

$$f(x) - f(a_0) \underset{V}{\frown} c_0, \quad f(x) - f(a_0) \in c_0(1 + \mathfrak{m})$$

for all $x \in V$, $x \geq a_1$ with $a_1 \in V$. The last condition is equivalent to $\left| \frac{f(x)-f(a_0)-c_0}{c_0} \right| < \epsilon_k$ for all k, where $1/\epsilon_k$ form a cofinal sequence in V. Each of these inequalities extends (by overspill) to an interval $[a_1, b_{1k}]$ with $b_{1k} > V$. Since $\mathcal{R}$ is α^+-saturated, there is an element $b_1 \in \mathcal{R}$ such that $1/\epsilon_k < b_1 < b_{1k}$ for all $k \in \mathbb{N}$. Clearly, $V < b_1$ and

$$f(x) - f(a_0) \in c_0(1 + \mathfrak{m}) \quad \text{for all} \quad x \in [a_1, b_1].$$

Next we get $f(x) - f(a_1) \in c_0 \cdot \mathfrak{m}$ for $x \in [a_1, b_1]$ whence, as before,

$$f(x) - f(a_1) \underset{V}{\sim} c_1 \in c_0 \cdot \mathfrak{m} \quad \text{and} \quad f(x) - f(a_1) \in c_1(1 + \mathfrak{m})$$

for all $x \in [a_2, b_2]$ with $a_2, b_2 \in \mathcal{R}$, $a_2 \in V$, $b_2 > V$.

By induction we can construct three sequences (a_n), (b_n), (c_n) of elements of $\mathcal{R}$ such that $a_n \in V$, $a_n > 1/\epsilon_n$ are cofinal in V, $b_n > V$, the sequence $(v(c_n))$ is strictly increasing, and

$$v\left(f(x) - f(a_n)\right) = v(c_n) \quad \text{for all} \quad x \in [a_{n+1}, b_{n+1}].$$

Since $\mathcal{R}$ is α^+-saturated, we can find an element $b \in \mathcal{R}$ such that $a_n < b < b_n$ for all $n \in \mathbb{N}$. Then $v\left(f(b) - f(a_n)\right) = v(c_n)$ is a strictly increasing sequence, and thus the valuation $v(g(x))$ of the function $g(x) := f(b) - f(x)$ does not stabilize. This contradiction with Proposition 1 completes the proof.

2. Valuation group of a simple extension.

Proposition 3. *Let $\mathcal{R}$ be a finitely generated and polynomially bounded structure, $V \subset \mathcal{R}$ a convex subring of $\mathcal{R}$, and $v : \mathcal{R} \setminus \{0\} \longrightarrow \Gamma = \Gamma_V$ the induced valuation. Then*

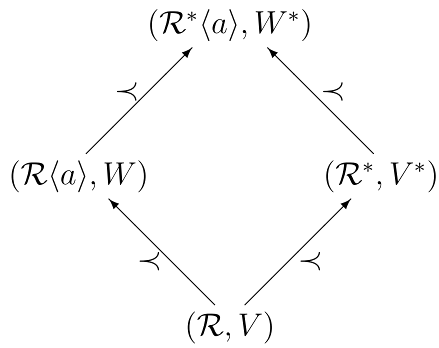
i) $\mathrm{rk}\,(v) \leq \dim_K \Gamma \leq \mathrm{rk}\,(\mathcal{R})$,

ii) If $(\mathcal{R}, V) \subset (\mathcal{R}\langle a \rangle, W)$, then $\dim_K \Gamma_W \leq \dim_K \Gamma + 1$*; more precisely,* $\Gamma_W = \Gamma \oplus K w(a)$ *whenever* $w(a) \notin \Gamma$.

We proceed with induction on $\mathrm{rk}\,\mathcal{R}$ and prove point ii), which is the induction step. By induction hypothesis, $\mathrm{rk}\,(v) =: d \leq \mathrm{rk}\,\mathcal{R} < \infty$. Then $\Gamma = \Gamma_V = \Gamma_1 \oplus \ldots \oplus \Gamma_d$, and we can find — as described in the observation of Section 1 — an elementary extension $(\mathcal{R}^*, V^*)$ of $(\mathcal{R}, V)$ such that

$$\Gamma^* = \underbrace{\mathbb{R} \oplus \ldots \oplus \mathbb{R}}_{d \text{ times}}.$$

Consider now an heir-coheir amalgam of elementary extensions



Clearly, $w^*(a) \notin \Gamma^*$ if $w(a) \notin \Gamma$. Were $w^*(a) \in \Gamma^*$, then

$$\exists\, c^* \in \mathcal{R}^* \ \ w^*(a) = w^*(c^*), \text{ i.e. } \frac{a}{c^*} \in W^* \setminus \mathfrak{m}_{W^*},$$

and thus, by the heir-coheir property,

$$\exists\, c \in \mathcal{R} \ \ w(a) = w(c), \text{ i.e. } \frac{a}{c} \in W \setminus \mathfrak{m}_W,$$

which is a contradiction.

It suffices to establish point ii) for the elementary extension

$$(\mathcal{R}^*, V^*) \subset (\mathcal{R}^*\langle a\rangle, W^*).$$

Indeed, let $b = f(a)$, where $f : \mathcal{R} \longrightarrow \mathcal{R}$ is a definable function. Suppose $w^*(b) \in \Gamma^* + \lambda w^*(a)$ for a $\lambda \in K$, whence

$$\exists\, c^* \in \mathcal{R}^* \ \ w^*(b) = w^*(c^* a^\lambda), \text{ i.e. } \frac{f(a)}{c^* a^\lambda} \in W^* \setminus \mathfrak{m}_{W^*}.$$

It follows from the heir-coheir property that

$$\exists\, c \in \mathcal{R} \ \ w(b) = w(c a^\lambda), \text{ i.e. } \frac{f(a)}{c a^\lambda} \in W \setminus \mathfrak{m}_W,$$

and thus $w(b) \in \Gamma + \lambda w(a)$, as required.

Now we shall show that if $w^*(a) \notin \Gamma^*$, then for every $b \in \mathcal{R}^*\langle a\rangle$ we have

$$w^*(b) \in \Gamma^* + Kw^*(a).$$

Since Γ^* is the direct sum of a finite number of copies of $\mathbb{R}$, and the ordered set $\mathbb{R}$ of real numbers is Dedekind complete, there exists an element $\gamma = v^*(r) \in \Gamma^*$, $r \in \mathcal{R}^*$, such that $w^*(a) - w^*(r)$ realizes the cut made by an isolated subgroup of Γ^* in Γ^*. For, if the cut made by $w^*(a)$ lies inside of an $\Gamma_i^* \simeq \mathbb{R}$, we take the real number $\delta \in \Gamma_i^*$ which is closest to $w^*(a)$. Hence $|\, w^*(a) - \delta| \, < (\Gamma_i^*)^+$, and either we are done or we repeat the reasoning for a Γ_j^* with $j < i$.

Therefore the element a/r realizes the cut made by a convex subring of $\mathcal{R}^*$ in $\mathcal{R}^*$. Clearly, $b = f(a/r)$ for some definable function $f : \mathcal{R}^* \longrightarrow \mathcal{R}^*$. It follows from the corollary to Proposition 1 that

$$w^*(b) = w^*(f(a/r)) = w^*(c(a/r)^\lambda) = w^*(c) - \lambda w^*(r) + \lambda w^*(a) \in \Gamma^* + Kw^*(a),$$

concluding the proof.

Corollary. *Consider a polynomially bounded, o-minimal theory T and a simple extension $(\mathcal{R}, V) \subset (\mathcal{R}\langle a\rangle, W)$ of models of the theory T_{conv}. Then* $\dim_K \Gamma_W \le \dim_K \Gamma_V + 1$*; more precisely,* $\Gamma_W = \Gamma_V \oplus Kw(a)$ *whenever* $w(a) \notin \Gamma_V$*.*

Indeed, if $b \in \mathcal{R}\langle a\rangle$, then $b \in \mathcal{R}'\langle a\rangle$ for a finitely generated substructure $\mathcal{R}' \prec \mathcal{R}$, $\mathrm{rk}\,(\mathcal{R}') < \infty$; let $V' := V \cap \mathcal{R}'$ and $\Gamma_{V'}$ be its valuation group. By Proposition 3, we get $w(b) \in \Gamma_{V'} + Kw(a) \subset \Gamma_V + Kw(a)$, as asserted.

Remark. Proposition 3 implies immediately a stronger inequality (cf. [1], Section 5):

If $\mathrm{rk}\,(\mathcal{R}) < \infty$ and $\bar{V}$ *is the residue field of the convex subring* V*, then*

$$\mathrm{rk}\,(\bar{V}) + \dim_K(\Gamma_V) \le \mathrm{rk}\,(\mathcal{R}).$$

From the above one can derive the following Wilkie inequality (*loc. cit.*) through an argument of Wilkie (cf. [12]), based on saturated models and an iteration procedure:

Suppose T is a polynomially bounded theory and $(\mathcal{R}, V) \prec (\mathcal{S}, W)$ are models of T_{conv} with $\mathrm{rk}\,(\mathcal{S}/\mathcal{R}) \le \infty$*. Then*

$$\dim_K(\Gamma_W/\Gamma_V) + \mathrm{rk}\,(\bar{W}/\bar{V}) \le \mathrm{rk}\,(\mathcal{S}/\mathcal{R}).$$

3. Valuation Property and Preparation Theorem.

Valuation Property. *Consider a polynomially bounded, o-minimal theory T and a simple extension*

$$(\mathcal{R}, V) \subset (\mathcal{R}\langle a \rangle, W)$$

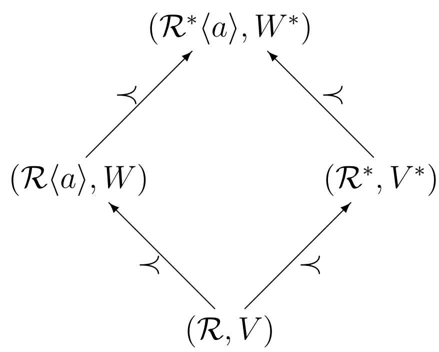
of models of the theory T_{conv} with valuation groups Γ_V and Γ_W, respectively. If $\Gamma_V \neq \Gamma_W$, then there is an $r \in \mathcal{R}$ for which $w(a-r) \notin \Gamma_V$.

We may assume that $\operatorname{rk} \mathcal{R} < \infty$, because $w(b) \notin \Gamma_V$ for some $b \in \mathcal{R}'\langle a \rangle$, where $\mathcal{R}'$ is a finitely generated substructure of $\mathcal{R}$. Now, if for some $r \in \mathcal{R}'$ the valuation $w'(a-r)$ does not belong to the valuation group $\Gamma_{V'}$ of the restriction $v' := v \mid \mathcal{R}'$, it follows from Proposition 3 that

$$w'(a-r) = \gamma + \lambda w'(b)$$

for some $\lambda \in K$, $\lambda \neq 0$, and $\gamma \in \Gamma_{V'}$. Hence $w(a-r) \notin \Gamma_V$.

Consider, as in the proof of Proposition 3, an heir-coheir amalgam of elementary extensions



Via the heir-coheir property, it suffices to establish the valuation property for the simple extension $(\mathcal{R}^*, V^*) \subset (\mathcal{R}^*\langle a \rangle, W^*)$. Indeed, $w^*(b) \notin \Gamma_{V^*}$, for otherwise

$$\exists\, r^* \in \mathcal{R}^* \quad w^*(b) = v^*(r^*),$$

and thus by the heir-coheir property we would get a contradiction

$$\exists\, r \in \mathcal{R} \;\; w(b) = v(r) \in \Gamma_V.$$

Further, if $w^*(a-r^*) \notin \Gamma_{W^*}$, then $w^*(a-r^*) \in \Gamma_{V^*} + \lambda w^*(b)$ for some $\lambda \in K$, $\lambda \neq 0$, whence

$$\exists\, r^*, s^* \in \mathcal{R}^* \;\; w^*(a-r^*) = v^*(s^*) + \lambda w^*(b).$$

Therefore, again by the heir-coheir property,

$$\exists\, r, s \in \mathcal{R} \;\; w(a-r) = v(s) + \lambda w(b) \notin \Gamma_V,$$

as required.

We are thus to show that if $w^*(b) \notin \Gamma_{V^*}$, then $w^*(a-r) \notin \Gamma_{V^*}$ for some $r \in \mathcal{R}^*$. Since the valuation group Γ_{V^*} is the direct sum of a finite number of copies of $\mathbb{R}$ (which are Dedekind complete), we deduce — similarly, as in the proof of Proposition 3 — that for some $v^*(r) \in \Gamma_{V^*}$, $r \in \mathcal{R}^*$, the element $w^*(b) - v^*(r)$ realizes the cut made by an isolated subgroup of Γ_{V^*} in Γ_{V^*}. Replacing b by b/r we may, of course, assume that b realizes the cut made by a convex subring of $\mathcal{R}^*$ in $\mathcal{R}^*$. Clearly, $a = f(b)$ for some definable function $f : \mathcal{R}^* \longrightarrow \mathcal{R}^*$ which is constant on no interval in $\mathcal{R}^*$. Now it follows from Proposition 2 that there exist $s \in \mathcal{R}^*$, $\lambda \in K$, $\lambda \neq 0$, and $c \in \mathcal{R}^*$ such that

$$w^*(a-r) = w^*(f(b)-r) = w^*(c\, b^{\lambda}) = w^*(c) + \lambda w^*(b) \notin \Gamma^*_{V^*},$$

which completes the proof.

The valuation property yields, via a routine compactness argument, the preparation theorem for one variable:

Corollary. *Let $\mathcal{R}$ be a polynomially bounded, o-minimal structure, $f : \mathcal{R} \longrightarrow \mathcal{R}$ be a definable function and $\epsilon \in \mathbb{Q}$, $\epsilon > 0$. Then there exist*

$$\lambda_1, \ldots, \lambda_k \in K, \quad r_1, \ldots, r_k, c_1, \ldots, c_k \in \mathcal{R}$$

such that for all $x \in \mathcal{R}$ we have

$$f(x) = |\, x - r_i|^{\lambda_i} \cdot c_i \cdot u$$

for an $i = 1, \ldots, k$ and some $u \in \mathcal{R}$ with $|\, u - 1| < \epsilon$.

Indeed, by passing to the theory T^{df} in the extended language $\mathcal{L}^{df}$, we shall deal with models $\mathcal{S}$ which are elementary extensions of $\mathcal{R}$, $\mathcal{R} \prec \mathcal{S}$. Through compactness, we must show that for each $a \in \mathcal{S}$ there exist $\lambda \in K$, $r, c \in \mathcal{R}$ and $u \in \mathcal{S}$, $|\,u - 1| < \epsilon$, such that

$$f(a) = |\,a - r|^{\lambda} \cdot c \cdot u.$$

Consider now, as the convex subrings of $\mathcal{R}$, $\mathcal{R}\langle a\rangle$ and $\mathcal{S}$, the convex hulls of the field of real numbers $\mathbb{R}$ in these fields. But the existence of $\lambda \in K$, $r, c \in \mathcal{R}$ and $u \in \mathcal{S}$, for which $w(u) = 0$ and $f(a) = |\,a - r|^{\lambda} \cdot c \cdot u$, follows immediately from the valuation property and the corollary to Proposition 3. Since $w(u) = 0$, u is of the form $u_0 + \mathbb{R}$-infinitesimal, and thus $|u/u_0 - 1| < \epsilon$. Replacing c and u with $u_0 c$ and u/u_0, respectively, we obtain the desired result.

The general version follows immediately through model-theoretic compactness and definable choice:

Preparation Theorem. *Under the previous assumptions, consider a definable function $f : \mathcal{R}^{n+1} \longrightarrow \mathcal{R}$ and an $\epsilon \in \mathbb{Q}$, $\epsilon > 0$. Then there exist $\lambda_1, \ldots, \lambda_k \in K$ and definable functions*

$$r_1, \ldots, r_k, c_1, \ldots, c_k : \mathcal{R}^n \longrightarrow \mathcal{R}, \quad u_1, \ldots, u_k : \mathcal{R}^{n+1} \longrightarrow (1 - \epsilon, 1 + \epsilon) \subset \mathcal{R}$$

such that for all $\bar{t} \in \mathcal{R}^n$ and $x \in \mathcal{R}$ we have

$$f(\bar{t}, x) = |\,x - r_i(\bar{t})|^{\lambda_i} \cdot c_i(\bar{t}) \cdot u_i(\bar{t}, x) \quad \text{for an} \;\; i = 1, \ldots, k.$$

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