n–real valuations and the higher level version of the Krull–Baer theorem[*,†]

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Abstract

In [Ci2] Cimprič gave examples of division rings containing an ordering of level $2m$ but not of level m for $m \in \mathbb{N}$. His examples were quite complicated. We give substantially simplified examples in Section 2. In Sections 3 and 4 we investigate this phenomenon using valuation theory. We define almost real and n–real valuations and study liftings of orderings from the residue division ring to the original division ring. Such liftings are not always possible (as is the case in the commutative setting), but we give a necessary and sufficient condition for a lifting to exist. We also prove a suitable generalization of the Baer–Krull theorem. Finally, in the last section we use our examples and the theory developed to answer a question given by Marshall & Zhang [MZ].

1. Introduction

Since Becker's breakthrough article on orderings of higher level [Be], this theory is being developed rapidly. It has been extended to division rings by Craven [Cr] and Powers [Po1] [Po2].

Some results extend straightforward from the commutative to the noncommutative case, while others fail to do so. One of such is the following result, observed already by Becker [Be, Korollar 2.3]:

Theorem 1: *Let K be a field. Then the following are equivalent:*

(i) $-1 \notin \sum K^2$,

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(ii) $-1 \notin \sum K^{2n}$ *for some* $n \in \mathbb{N}$,
(iii) $-1 \notin \sum K^{2n}$ *for all* $n \in \mathbb{N}$.

It was shown in Cimprič [Ci2] that this result fails in the noncommutative setting. He gave examples of division rings having orderings of level $2m-2$ but not of level $m-1$. His examples were quite complicated, so we present here more elementary examples, which turn out to be division rings of fractions of skew polynomial rings in 2 variables. Later in Section 3 we develop valuation theory to study sums of permuted products of n–th powers in division rings. We also develop valuation theory to study liftings of orderings from residue division rings to original division rings. In commutative case such liftings always exist. It is even possible to classify all of them. This is the so–called Baer–Krull theorem. For general division rings liftings need not exist. However, we are able to give a necessary and sufficient condition for a lifting to exist. We also get an appropriate generalization of the higher level version of the Baer–Krull theorem. In the fourth section we return to the examples given in Section 2 to apply the developed valuation theory. We also answer a question concerning real places given by Marshall & Zhang [MZ]; see Section 5.

2. Examples

Let A be an associative ring with identity. A semigroup homomorphism $\sigma : (A, \cdot) \longrightarrow (\mathbb{C}, \cdot)$ is called a *signature* provided the following hold:

(S_1) $\sigma(-1) = -1$,
(S_2) $\sigma^{-1}(0) = \{0\}$,
(S_3) for some $n \in \mathbb{N}$ and each $x \in A \setminus \{0\}$: $\sigma(x)^n = 1$,
(S_4) $\sigma^{-1}\big(\{0,1\}\big) =: P_\sigma$ is closed under addition.

The exponent n in (S_3) is called an *exponent* of σ and $n/2$ is called a *level* of σ. If n is the smallest natural number satisfying (S_3), then n is the *strict exponent* of σ and $n/2$ is called the *strict level* of σ. The set P_σ is called an *ordering* of level $n/2$ (or exponent n). Note that exponents of signatures are always even and any ring with a signature is a domain.

By the Artin–Schreier theorem a division ring D contains an ordering of level n iff $-1 \notin \sum D^{2n}$, where $\sum D^{2n}$ denotes the sum of permuted $2n$–th powers of elements from D. The commutative version was given by Becker [Be, Satz 1.4 and Satz 2.17], the noncommutative by Powers [Po1, Theorem 3.13].

We now present the examples announced above. Throughout this section k will denote a formally real field and $R_\ell := k[X][Y;\sigma_\ell]$, where σ_ℓ is the endomorphism of $k[X]$ fixing k pointwise and sending $X \longmapsto -X^{2\ell+1}$ for $\ell \in \mathbb{N}$. It is known that R_ℓ is a left Ore domain but is not a right Ore domain, see e.g. [La, §9 and §10].

Proposition 2: *There does not exist an ordering of level* ℓ *of* R_ℓ.

Proof. It is enough to prove that $-\sum R_\ell^{2\ell} \cap \sum R_\ell^{2\ell} \neq \{0\}$. Obviously, $X^{2\ell-1}YXY^{2\ell-1} \in \sum R_\ell^{2\ell}$. On the other hand, $X^{2\ell-1}YXY^{2\ell-1} = -X^{4\ell}Y^{2\ell} \in -\sum R_\ell^{2\ell}$. Since R_ℓ is a domain, this concludes the proof. ■

Proposition 3: *There exists an ordering of strict level 2ℓ of R_ℓ.*

Proof. Every element of R_ℓ can be written uniquely as

$$f = \sum_{m,n} a_{m,n} X^m Y^n.$$

We order the monomials X^mY^n anti–lexicographically, i.e.,

$$X^{m_1}Y^{n_1} \succ X^{m_2}Y^{n_2} \;:\Longleftrightarrow\; n_1 > n_2 \vee (n_1 = n_2 \wedge m_1 > m_2).$$

From now on we fix an ordering of k and for $0 \neq f \in R_\ell$ define

$$\sigma(f) := \operatorname{sign}(a_{m,n}) \exp\left(\frac{\pi i m}{2\ell}\right),$$

if $f = a_{m,n}X^mY^n+$ (lower terms). Moreover, put $\sigma(0) := 0$. We claim that σ is a signature of strict level 2ℓ. Note that $\sigma(-1) = -1$ and $\sigma^{-1}(0) = \{0\}$. To show that σ is multiplicative, take $0 \neq g = b_{p,q}X^pY^q+$ (lower terms) $\in R_\ell$. Then $fg = a_{m,n}b_{p,q}X^mY^nX^pY^q+$ (lower terms). Since

$$X^mY^nX^pY^q = (-1)^{np}X^{m+(2\ell+1)^n\cdot p}Y^{n+q},$$

we have $\sigma(fg) = \operatorname{sign}\left((-1)^{np}a_{m,n}b_{p,q}\right)\exp\left(\frac{\pi i}{2\ell}\left(m + p(2\ell+1)^n\right)\right)$. Note that $(2\ell+1)^n = \sum_{j=0}^{n}\binom{n}{j}(2\ell)^j$. Hence $\exp\left(\frac{\pi i}{2\ell}p(2\ell+1)^n\right) = \exp\left(\frac{\pi i}{2\ell}p + \pi i n p\right) = \exp\left(\frac{\pi i}{2\ell}p\right)(-1)^{np}$. This gives

$$\sigma(fg) = \operatorname{sign}(a_{m,n})\operatorname{sign}(b_{p,q})\exp\left(\frac{\pi i}{2\ell}p\right)\exp\left(\frac{\pi i}{2\ell}m\right) = \sigma(f)\sigma(g).$$

As σ is multiplicative, it is easy to see that $\sigma(f)^{4\ell} = 1$ for $0 \neq f \in R_\ell$.

Now assume $\sigma(f) = \sigma(g) = 1$. If the highest monomials of f and g differ, then $\sigma(f+g) = 1$ since the leading term of $f+g$ is either the leading term of f or of g. If the highest monomials are the same, say X^mY^n, then $\operatorname{sign}(a_{m,n}) = \operatorname{sign}(b_{m,n})$ and thus $\operatorname{sign}(a_{m,n} + b_{m,n}) = \operatorname{sign}(a_{m,n}) = \operatorname{sign}(b_{m,n})$. This proves $\sigma(f+g) = 1$.

To finish the proof, note that $\sigma(X) = \exp\left(\frac{\pi i}{2\ell}\right)$ and thus $\sigma(X)^j \neq 1$ for $1 \leqslant j < 4\ell$. ■

Theorem 4: *For each $\ell \in \mathbb{N}$ there exists a division ring D_ℓ which is the division ring of left fractions of the skew polynomial ring R_ℓ in 2 variables, such that D_ℓ does not contain an ordering of level ℓ, but contains an ordering of strict level 2ℓ.*

Proof. This is clear since by Cimprič [Ci1, Proposition 5.2], every signature of R_ℓ extends (uniquely) to a signature of D_ℓ. ■

3. n–real valuations and higher level Krull–Baer theorem

Throughout this section let D denote a division ring and $D^\times$ its group of units. Let $\prod_n D^\times$ denote the subgroup of $D^\times$ generated by n–th powers and multiplicative commutators.

All valuations considered will be invariant. A valuation with (not necessarily commutative) value group Γ_v will be denoted by $v\colon D \longrightarrow \Gamma_v \cup \{0\}$ and $\mathcal{O}_v, \mathfrak{m}_v, \mathrm{k}_v, \Gamma_v$ will represent its valuation ring, its maximal ideal, its residue division ring and its value group respectively (see Schilling [Sch] for more details). For $a \in \mathcal{O}_v$, $\overline{a} := a + \mathfrak{m}_v$ will denote its image in k_v.

A set $P \subseteq D$ is a *complete preordering of exponent* n or *level* $n/2$ if

$$\prod_n D^\times \subseteq P, \quad -1 \notin P, \quad P + P \subseteq P, \quad P \cdot P \subseteq P, \quad a^2 \in P \;\Rightarrow\; a \in P \cup -P.$$

Note that an ordering of level n is also a complete preordering of level n and every complete preordering of level n can be extended to an ordering of level n (see e.g. [Po1, Theorem 3.13]). Exponents of complete preorderings are always even.

A valuation v is said to be *compatible* with a complete preordering P if $1 + \mathfrak{m}_v \subseteq P$. From [Po1, Theorem 3.13] it follows that

$$\sum D^n = \bigcap \{P \mid P \text{ a complete preordering of exponent } n \text{ of } D\}$$

and for a given complete preordering P there exists a valuation v with its valuation ring $\mathcal{O}_v = A(P) := \{a \in D \mid \exists r \in \mathbb{Q}_{>0} : r \pm a \in P\}$ and maximal ideal $\mathfrak{m}_v = I(P) := \{a \in D \mid \forall r \in \mathbb{Q}_{>0} : r \pm a \in P\}$, that is compatible with P and induces an ordering $\overline{P} = \overline{P \cap \mathcal{O}_v}$ of level 1 of k_v (see e.g. [Po1] for more details).

Remark: If the preordering has exponent $2n$, the induced ordering of k_v has the property $\overline{\mathcal{O}_v^\times \cap \prod_{2n} D^\times} \subseteq \overline{P}$.

Definition: The valuation v is *n–real* if k_v admits an ordering $\overline{P}$ of level 1 with the property $\overline{\mathcal{O}_v^\times \cap \prod_{2n} D^\times} \subseteq \overline{P}$. An ordering $\overline{P}$ of k_v will be called *n–compatible with* v if $\overline{\mathcal{O}_v^\times \cap \prod_{2n} D^\times} \subseteq \overline{P}$. 1–real valuations will be called *real valuations.* A valuation v is called *almost real* if k_v is formally real.

Remark: Note that the existence of a level 1 ordering $\overline{P}$ of k_v which is n–compatible with v implies that v is n–real.

As we will show, every ordering $\overline{P}$ of level 1 of the residue division ring k_v which is n–compatible with v can be lifted to a complete preordering of level n of D. From last remarks it follows that the condition "v is n–real" is also necessary for lifting. But as a contrast to the commutative theory not every n–real valuation is also m–real for $m < n$. For example, the division ring of fractions D_ℓ from Section 2 admits level 2ℓ orderings but it doesn't admit level ℓ orderings. Therefore there exist 2ℓ–real valuations but no ℓ–real valuations on D_ℓ. This is the reason why Theorem 1 fails for division rings.

In the commutative setting every almost real valuation is n–real for each n. This fails in the noncommutative case. In Section 4 we will give an example of a division ring containing no orderings of any level, but admitting an almost real valuation.

For the study of sums of permuted products of n–th powers it is more suitable to deal with orderings and preorderings defined by a subset $P \subseteq D$ that satisfies certain conditions than with signatures. This enables us to give a constructive proof of Theorem 6 which follows Becker's original construction [Be, Satz 2.4] for commutative fields.

Lemma 5: *Let D be a division ring and $v\colon D \twoheadrightarrow \Gamma \cup \{0\}$ a valuation on D, where Γ^{ab} is without n-torsion (e.g. Γ is abelian or $[\Gamma,\Gamma]$ is relatively convex in Γ). Then v induces a surjective homomorphism $\overline{v}\colon D^\times/\prod_n D^\times \longrightarrow \Gamma/\prod_n \Gamma$ with a section $\mu\colon \Gamma/\prod_n \Gamma \longrightarrow D^\times/\prod_n D^\times$, i.e. $\overline{v}\circ\mu = \mathrm{id}$.*

Proof. The existence of $\overline{v}\colon D^\times/\prod_n D^\times \longrightarrow \Gamma/\prod_n \Gamma$ is obvious. Let $[D^\times, D^\times]$ denote the normal subgroup generated by multiplicative commutators in $D^\times$. By definition, $[D^\times, D^\times] \subseteq \prod_n D^\times$. For a group G denote $G^{\mathrm{ab}} := G/[G,G]$. It is easy to see that $\prod_n G/[G,G] \cong (G^{\mathrm{ab}})^n$. Hence by isomorphism theorem, $D^\times/\prod_n D^\times \cong {D^\times}^{\mathrm{ab}}/({D^\times}^{\mathrm{ab}})^n$ and $\Gamma/\prod_n \Gamma \cong \Gamma^{\mathrm{ab}}/(\Gamma^{\mathrm{ab}})^n$, so $\overline{v}$ induces $\tilde{v}\colon {D^\times}^{\mathrm{ab}}/({D^\times}^{\mathrm{ab}})^n \longrightarrow \Gamma^{\mathrm{ab}}/(\Gamma^{\mathrm{ab}})^n$. Using [Be, Lemma 2.3] we get a section $\tilde{\mu}$ for $\tilde{v}$, that induces a section $\mu\colon \Gamma/\prod_n \Gamma \longrightarrow D^\times/\prod_n D^\times$ for $\overline{v}$. ∎

Remark: If n is prime or its prime factorization $n = \prod_i p_i$ consists of pairwise different primes, i.e. $p_i \neq p_j$ for $i \neq j$, then there exists a section $\mu\colon \Gamma/\prod_n \Gamma \longrightarrow D^\times/\prod_n D^\times$. Namely, $A/\prod_n A \cong \prod_i \left(A/\prod_{p_i} A\right)$ for every group A, so it is enough to prove the existence of sections $\mu_i\colon \Gamma/\prod_{p_i}\Gamma \longrightarrow D^\times/\prod_{p_i} D^\times$ of homomorphisms $\overline{v}_i\colon D^\times/\prod_{p_i} D^\times \longrightarrow \Gamma/\prod_{p_i}\Gamma$. Since both abelian groups $D^\times/\prod_{p_i} D^\times$ and $\Gamma/\prod_{p_i}\Gamma$ are $\mathbb{Z}/p_i\mathbb{Z}$–vector spaces, sections μ_i exist.

Unfortunately we were unable to prove the existence of sections in general.

Theorem 6: *Assume $n \in \mathbb{N}$ is even, $v : D \twoheadrightarrow \Gamma \cup \{0\}$ is $\frac{n}{2}$–real and let $\overline{v}\colon D^\times/\prod_n D^\times \longrightarrow \Gamma/\prod_n \Gamma$ be the induced group homomorphism. Let Γ be such that $\overline{v}$ admits a section $\mu\colon \Gamma/\prod_n \Gamma \longrightarrow D^\times/\prod_n D^\times$ as in Lemma 5. Fix a set of representatives $\mathfrak{U} \subseteq D^\times$ for $\mu(\Gamma/\prod_n \Gamma)$ with $1 \in \mathfrak{U}$.*

Let $\overline{P}$ be a level 1 ordering of k_v that is n–compatible with v. Suppose Γ_0 is a subgroup of Γ containing $\prod_n \Gamma$ such that the Sylow 2–subgroup of Γ/Γ_0 is cyclic of order 2^r, $r \geqslant 0$ and $\chi\colon \Gamma_0 \longrightarrow \mathrm{k}_v^\times/\overline{P}^\times$ is a character satisfying $\chi(\prod_n \Gamma) = 1$ and $\chi(\Gamma_0 \cap \prod_{2^r}\Gamma) \neq 1$ if $r \geqslant 1$.

Define $\mathfrak{U}_0 \subseteq \mathfrak{U}$ to be the system of representatives of $\mu(\Gamma_0/\prod_n \Gamma)$ and for every $a \in \mathfrak{U}_0$ denote $M_a := \left\{\varepsilon \in \mathcal{O}_v^\times \,|\, \chi(v(a)) = \overline{\varepsilon}\overline{P}\right\}$. Then:

(a) $P := \bigcup_{a\in\mathfrak{U}_0} aM_a \prod_n D^\times$ *is a complete preordering of exponent* n *compatible with* v *and induces the given ordering* $\overline{P}$ *of* k_v.

(b) *Every complete preordering of exponent* n *compatible with* v *that induces the ordering* $\overline{P}$ *of* k_v *is obtained in this way.*

Proof. It is easy to see that $\mathfrak{U}$ satisfies following properties:

(1) for every $\alpha \in \Gamma$ there exists a unique $a \in \mathfrak{U}$ such that $\alpha \prod_n \Gamma = v(a) \prod_n \Gamma$;

(2) for every $a, b \in \mathfrak{U}$ there exist $c \in \mathfrak{U}$ and $p \in \prod_n D^\times$ satisfying $ab = cp$.

Some remarks on the system of representatives:

- From property (1) it follows that for every $x \in D^\times$ there exist $\varepsilon \in \mathcal{O}_v^\times$, $y \in \prod_n D^\times$ and a unique $a \in \mathfrak{U}$ such that $x = a\varepsilon y$.
- If for some $a \in \mathfrak{U}$ there exists $t \in \mathbb{N}$ satisfying $v(a)^{2^t} \in \Gamma_0$, the assumptions on Γ_0 imply $v(a)^{2^r} \in \Gamma_0$.
- From property (2) we conclude that $a^{2^r} = bx$ for some $x \in \prod_n D^\times$ and $b \in \mathfrak{U}$. Properties of Γ_0 imply $v(b) \prod_n \Gamma = v(a_0) \prod_n \Gamma$ for an $a_0 \in \mathfrak{U}_0$. Therefore by property (1), $b = a_0$. Under these assumptions:

$$a^{2^r} = a_0 x \text{ for an } x \in \prod\nolimits_n D^\times, \text{ where } a_0 \in \mathfrak{U}_0.$$

(a) We have $M_1 = \{\varepsilon \in \mathcal{O}_v^\times \,|\, \overline{\varepsilon} \in \overline{P}^\times\}$, thus $\prod_n D^\times \subseteq P$ and P is compatible with v, if we prove that P is a complete preordering.

Obviously $\overline{P} = \overline{M_1}$. Since $M_1 \subseteq P$, also $\overline{M_1} \subseteq \overline{\mathcal{O}_v^\times \cap P}$. For the reverse inclusion take $a \in \mathcal{O}_v^\times \cap P$ and write $a = \tilde{a}\eta x$ for $x \in \prod_n D^\times$, $\eta \in M_{\tilde{a}}$ and $\tilde{a} \in \mathfrak{U}_0$. Since $\tilde{a}$ is uniquely determined by a and $v(a) \prod_n \Gamma = v(\tilde{a}) \prod_n \Gamma = \prod_n \Gamma$, it follows that $\tilde{a} = 1$ and so $\eta \in M_1$. We conclude that $a = \eta x$, where $x \in \mathcal{O}_v^\times \cap \prod_n D^\times$. Because v is $\frac{n}{2}$–real, $\overline{x} \in \overline{\mathcal{O}_v^\times \cap \prod_n D^\times} \subseteq \overline{P}$. Hence $\overline{a} \in \overline{M_1 P} = \overline{P}$.

Choose $a, b \in \mathfrak{U}_0$, $\varepsilon \in M_a$, $\eta \in M_b$ and $x, y \in \prod_n D^\times$. We want to prove that $t := a\varepsilon x + b\eta y \in P$ and $s := ab\varepsilon\eta z \in P$, where $z = xyc$ and c is a product of commutators. If $v(a\varepsilon x) \neq v(b\eta y)$, we may w.l.o.g. assume that $v(a\varepsilon x) > v(b\eta y)$. Then $b\eta y = a\varepsilon x\psi$ for some $\psi \in \mathfrak{m}_v$. It follows that $t = a\varepsilon x(1+\psi) = a\varepsilon(1+\psi)\tilde{x}$, where $x \in \prod_n D^\times$. Because $\varepsilon(1+\psi) \in M_a \cdot M_1 \subseteq M_a$, we have $t \in P$. If $v(a\varepsilon x) = v(b\eta y)$, then $a = b$ by the uniqueness property. Hence, $yx^{-1} \in \mathcal{O}_v^\times \cap \prod_n D^\times \subseteq \mathcal{O}_v^\times \cap P$, which gives $\overline{yx^{-1}} \in \overline{P}$. We conclude that $\eta yx^{-1} \in M_a$, so t can be written in the form $t = a(\varepsilon + \eta yx^{-1})x$. Since the classes $\overline{\varpi}\overline{P}^\times$ are additively closed, it follows that $M_a + M_a \subseteq M_a$. Therefore $t \in P$.

We have $s = ab\varepsilon\eta z$, where $z \in \prod_n D^\times$. By property (2) this gives $ab = cp$ for some $p \in \prod_n D^\times$ and $c \in \mathfrak{U}_0$. It follows $s = c\varepsilon\eta z'$ for some $z' \in \prod_n D^\times$. It suffices to prove that $\varepsilon\eta \in M_c$, i.e. $\overline{\varepsilon\eta}\overline{P} = \chi(v(c))$. Since $v(a)v(b) = v(c)v(p)$, $v(p) \in \prod_n \Gamma$ and $\chi(\prod_n \Gamma) = \overline{P}^\times$, the assertion follows.

Suppose that $-1 = a\varepsilon x$, where $a \in \mathfrak{U}_0, \varepsilon \in \mathcal{O}_v^\times$ and $x \in \prod_n D^\times$. Then $\prod_n \Gamma = v(-1)\prod_n \Gamma = v(a)\prod_n \Gamma$, therefore by the uniqueness property $a = 1$. It follows that $-1 = \varepsilon x$, so $x \in \mathcal{O}_v^\times \cap P$ and hence $-1 \in \overline{P}$, a contradiction.

From the above it follows that P is a preordering of exponent n. All that remains to be seen is that P is complete. From the construction, $v(P^\times) = \Gamma_0$. We conclude $v^{-1}(\Gamma_0) = P^\times \cup -P^\times$, since for every $\varepsilon \in \mathcal{O}_v^\times$ also $\varepsilon \in P \cup -P$, as $\overline{P}$ is an ordering of level 1. Take $x \in D^\times$ satisfying $x^2 \in P$ and $x \notin P \cup -P$. Then $v(x)^2 \in \Gamma_0$ and $v(x) \notin \Gamma_0$. The 2–Sylow subgroup of Γ/Γ_0 is cyclic of order $2^r \geqslant 2$. Choose $w \in D^\times$ such that $\omega := v(w)$ satisfies $\omega^{2^r} \in \Gamma_0$ and $\omega^{2^{r-1}} \notin \Gamma_0$. Therefore $v(x)\Gamma_0 = \omega^{2^{r-1}}\Gamma_0$, thus $x = w^{2^{r-1}}x_1$, where $x_1 \in P \cup -P$ and so $w^{2^{r-1}} \notin P \cup -P$ and $w^{2^r} \in P$. Write $w^{2^{r-1}} = a\varepsilon y$ for $a \in \mathfrak{U}, \varepsilon \in \mathcal{O}_v^\times$ and $y \in \prod_n D^\times$. Then $w^{2^r} = a^2\varepsilon^2 z$ for some $z \in \prod_n D^\times$; note that $a^2\varepsilon^2 z \in P$. Because of property (2) there exists $b \in \mathfrak{U}$ such that $a^2 = b\tilde{z}$, $\tilde{z} \in \prod_n D^\times$ and hence $w^{2^r} = b\varepsilon^2 z' \in P$ for some $z' \in \prod_n D^\times$. By the definition of P, M_b and uniqueness, $b \in \mathfrak{U}_0$ and $\chi(v(b)) = \overline{\varepsilon}^2\overline{P} = \overline{P}$. Also: $\chi(\omega^{2^r}) = \chi(v(w)^{2^r}) = \chi(v(b)) = \overline{P}$. Since $\Gamma_0 \cap \prod_{2^r}\Gamma$ is generated by $\prod_{2^r}\Gamma_0$ and ω^{2^r}, $\chi(\prod_{2^r}\Gamma_0) = \overline{P}$ and $\chi(\Gamma_0 \cap \prod_{2^r}\Gamma) \neq \overline{P}$ by definition, we get $\chi(\omega^{2^r}) = -\overline{P}$, a contradiction.

(b) Suppose that P is a complete preordering of exponent n of D compatible with v and induces the ordering $\overline{P} = \overline{\mathcal{O}_v \cap P}$ of level 1 of k_v. Define $\Gamma_0 := v(P^\times)$ and the character $\chi\colon \Gamma_0 \longrightarrow \mathrm{k}_v^\times/\overline{P}^\times$ as $\chi(v(u)) = \overline{\varepsilon}\overline{P}^\times$, where $u = a\varepsilon x$ for some $a \in \mathfrak{U}$, $\varepsilon \in \mathcal{O}_v^\times$ and $x \in \prod_n D^\times$. Clearly $\prod_n \Gamma \subseteq \Gamma_0$. Let w be a permuted product of n–th powers. Obviously $\chi(v(w)) = \overline{P}$, therefore $\chi(\prod_n \Gamma) = \overline{P}$. The condition $x^2 \in P \Rightarrow x \in P \cup -P$ implies that the Sylow 2–subgroup of $D^\times/P^\times$ is cyclic (see [Wa, 1.4.1 Proposition]). Since the Sylow 2–subgroup of Γ/Γ_0 is its epimorphic image, we conclude that the Sylow 2–subgroup of Γ/Γ_0 is cyclic of finite order 2^r. Choose $w \in D^\times$ such that $v(w)^{2^{r-1}} \notin \Gamma_0$ and $v(w)^{2^r} \in \Gamma_0$.

We want to prove that $\chi(v(w)^{2^r}) = -\overline{P}$. Equivalently, if we write $w^{2^r} = a\varepsilon x$, where $a \in \mathfrak{U}_0$, $\varepsilon \in \mathcal{O}_v^\times$ and $x \in \prod_n D^\times$, then we have to show that $\overline{\varepsilon} \in -\overline{P}$. If $\overline{\varepsilon} \in \overline{P}$, then $v(w^{2^r}) = v(u)$ for some $u \in P^\times$, where $u = a\eta x$, $a \in \mathfrak{U}_0$ and $\overline{\eta} \in \overline{P}$. We conclude that $\eta \in P$ and so $a \in P$. Let $w^{2^{r-1}} = b\tau y$. Then $w^{2^r} = b^2\tau^2 z = a\tau^2 z'$ because of the properties (1) and (2) of the system $\mathfrak{U}_0$. From $a \in P$ and $\tau^2 \in P$ ($\overline{P}$ is an ordering of level 1) it follows that $w^{2^r} \in P$ and thus $w^{2^{r-1}} \in P \cup -P$. In this case $v(w)^{2^{r-1}} \in \Gamma_0$, a contradiction.

Let $u \in P$. It can be written in the form $u = a\varepsilon x$, $a \in \mathfrak{U}_0$, $x \in \prod_n D^\times$ and $\varepsilon \in \mathcal{O}_v^\times$. From the definition of χ and from the fact that $\prod_n \Gamma \subseteq \ker\chi$ it follows that $\chi(v(a)) = \chi(v(u)) = \overline{\varepsilon}\overline{P}^\times$. Hence $P \subseteq \bigcup_{a \in \mathfrak{U}_0} aM_a \prod_n D^\times$.

Take $a \in \mathfrak{U}_0$ and $\varepsilon \in M_a$. There exists $\eta \in \mathcal{O}_v^\times$ with the property $a\eta \in P$. Also $\overline{\eta}\overline{P}^\times = \chi(v(a)) = \overline{\varepsilon}\overline{P}^\times$. Then $\overline{\varepsilon} = \overline{\eta p}$ for some $\overline{p} \in \overline{P}^\times$. Hence $\varepsilon \in \eta p + \mathfrak{m}_v = \eta p(1 + \mathfrak{m}_v)$. Since $p \in \mathcal{O}_v^\times \cap P$, we have $a\varepsilon \in a\eta p(1 + \mathfrak{m}_v) \subseteq P$, which completes the proof. ■

To get at least some information about orderings compatible with v that induce a fixed level 1 ordering of k_v, we will revert to signatures. The short exact sequence from Becker and Rosenberg [BR, Theorem 2.6] that enables the classification of all pullbacks of a given signature of k_v, is no longer split exact. Nevertheless, given a n–real valuation v and a signature π of k_v of level 1, which is n–compatible with v, we will construct all pullbacks of π of level n with respect to v. In other words, given a level n ordering of D compatible with v that induces the level 1 ordering $\ker \pi \cup \{0\}$ of k_v, we will be able to get all level n orderings of D that are compatible with v and induce the same level 1 ordering of k_v.

The definition of a level n signature of a division ring D simplifies a bit in comparison to signatures on general rings:

Definition: Let $\mu = \{z \in \mathbb{C} \mid z^r = 1 \text{ for some } r \in \mathbb{N}\}$. For any group G we write $G^* = \mathrm{Hom}(G, \mu)$. An element $\chi \in (D^\times)^*$ is called a *signature of level n* if $\ker \chi$ is additively closed and $\prod_{2n} D^\times \subseteq \ker \chi$. The set $\{0\} \cup \ker \chi$ is a level n ordering of D.

Given a valuation v on D, we say that a character $\chi \in (D^\times)^*$ is compatible with v, in symbols $\chi \sim v$ or $\chi \sim \mathcal{O}_v$, if $1 + \mathfrak{m}_v \subseteq \ker \chi$. Note that each character $\chi \sim v$ yields a character $i^*(\chi) \in (k_v^\times)^*$ called the *pushdown* of χ (with respect to v). Signatures χ of D whose pushdowns are π are called *pullbacks* of π. A signature π of k_v satisfying $\overline{\mathcal{O}_v^\times \cap \prod_{2n} D^\times} \subseteq \ker \pi$ will be called *n–compatible with v*.

The connection between a signature and its pushdown is the same as in commutative case (see Becker, Harman and Rosenberg [BHR, Lemma 1.10 and Proposition 2.5]).

Lemma 7: *Let v be a valuation on D and $\pi \in (k_v^\times)^*$. For any $\chi \in (D^\times)^*$ the following are equivalent:*

(i) *$\chi \sim v$ and $i^*(\chi) = \pi$.*
(ii) *For all $a \in \mathcal{O}_v^\times$, we have $\chi(a) = \pi(\overline{a})$.*

Proposition 8: *Let v be a valuation on D and χ a character with $\chi \sim v$. Then $\ker \chi$ is additively closed iff $\ker i^*(\chi)$ is additively closed.*

Remark: If χ from Proposition 8 is a signature of level n of D, then $i^*(\chi)$ is a signature of level n of k_v satisfying $\overline{\mathcal{O}_v^\times \cap \prod_{2n} D^\times} \subseteq \ker i^*(\chi)$.

The following theorem will describe all signatures of level n compatible with a given n–real valuation, that are all pullbacks of a given signature π of level n of the residue division ring which is n-compatible with v, relative to a chosen pullback of π. The theorem is the same as in commutative case (see [BR, Theorem 2.6] and [BHR, Theorem 1.12]).

Theorem 9: *Let v be a valuation on D and let π be a signature of level n of k_v which is n–compatible with v. Then:*

(1) *π has a pullback of level n with respect to v.*

(2) *If η is a fixed pullback of π with respect to v of level n, then all other pullbacks of π with respect to v of level n are given by $\eta\cdot(\tau\circ v)$ with τ running through $(\Gamma_v/\prod_{2n}\Gamma_v)^*$.*

Proof. (1) Proof was done in [Po1, Lemma 3.11].

(2) Let χ be an arbitrary pullback of π with respect to v of level n. According to Lemma 7, we have $\chi=\eta$ on $\mathcal{O}_v^\times$. Hence, $\eta^{-1}\cdot\chi\in(D^\times/\mathcal{O}_v^\times)^*$. Since $v\colon D^\times/\mathcal{O}_v^\times\longrightarrow\Gamma_v$ is an isomorphism, there exists a unique $\tau\in\Gamma^*$ such that $\eta^{-1}\cdot\chi=\tau\circ v$. In other words, $\chi=\eta\cdot(\tau\circ v)$. Since $\prod_{2n}D^\times\subseteq\ker(\eta^{-1}\cdot\chi)$, we have $\prod_{2n}\Gamma_v=v(\prod_{2n}D^\times)\subseteq\ker\tau$, so τ can be considered as an element of $(\Gamma_v/\prod_{2n}\Gamma_v)^*$.

Let $\chi=\eta\cdot(\tau\circ v)$ for some $\tau\in(\Gamma_v/\prod_{2n}\Gamma_v)^*$. Obviously $\prod_{2n}D^\times\subseteq\ker\chi$, $\chi\in(D^\times)^*$ and $1+\mathfrak{m}_v\subseteq\ker\chi$. Hence χ is a signature of level n compatible with v. Also $\chi(a)=\eta(a)=\pi(\overline{a})$ for all $a\in\mathcal{O}_v^\times$. This completes the proof. ∎

Let $\mathrm{Sgn}^n(D)$ denote the set of all signatures of level n of a division ring D and set $\mathrm{Sgn}_v^n(D):=\{\chi\in\mathrm{Sgn}^n(D)\mid\chi\sim v\}$. In addition let $\overline{\mathrm{Sgn}}_v^n(\mathrm{k}_v):=\{\chi\in\mathrm{Sgn}^n(\mathrm{k}_v)\mid\overline{\mathcal{O}_v^\times\cap\prod_{2n}D^\times}\subseteq\ker\chi\}$ and $\mathrm{Sgn}(D):=\bigcup_n\mathrm{Sgn}^n(D)$.

Theorem 10: *There exists a (non–canonical) bijection*

$$\mathrm{Sgn}_v^n(D)\longrightarrow\overline{\mathrm{Sgn}}_v^n(\mathrm{k}_v)\times\left(\Gamma_v/\prod_{2n}\Gamma_v\right)^*$$

Proof. By Theorem 9 (1) for every $\chi\in\overline{\mathrm{Sgn}}_v^n(\mathrm{k}_v)$ there exists a pullback. For every $\chi\in\overline{\mathrm{Sgn}}_v^n(\mathrm{k}_v)$ choose a pullback ψ_χ. According to Theorem 9 (2) all other pullbacks are of the form $\theta=\psi_\chi\cdot(\tau_\theta\circ v)$ with τ_θ running through $(\Gamma_v/\prod_{2n}\Gamma_v)^*$. Now the mapping is given by $\theta\mapsto(\overline{\theta},\tau_\theta)$ and is clearly a bijection. ∎

The power of this theorem lies in the fact that for every signature χ of D there is a compatible valuation ring $A(\chi)$, so that the pushdown of χ is a signature of an archimedean ordering of k_v. This enables us to describe the elements of $\mathrm{Sgn}^n(D)$ as in [BHR, p. 60].

Corollary 11: *All elements of $\mathrm{Sgn}^n(D)$ can be described according to the following procedure:*

(1) *Determine all n–real valuation rings $\mathcal{O}_v$ such that k_v has an archimedean ordering P which is n–compatible with v.*
(2) *Given $(\mathcal{O}_v,\mathrm{k}_v,P)$ as in (1), choose a signature $\eta\sim v$ of level n with pushdown $i^*(\eta)=\mathrm{sgn}_P$.*
(3) *Set $\chi=\eta\cdot(\tau\circ v)$, where v is a valuation corresponding to $\mathcal{O}_v$ and τ runs through the character group of $\Gamma_v/\prod_{2n}\Gamma_v$.*

The above results give a shorter proof of the classification of all signatures of the ring of quantum polynomials as found in [Ci3, Theorem 14]. First let us recall some basic definitions.

For every $1 \neq q \in \mathbb{R}_{>0}$ we define the ring $A_q := \mathbb{R}\langle X,Y\rangle/I_q$, where $\mathbb{R}\langle X,Y\rangle$ is the free associative algebra on $\{X,Y\}$ and I_q is the principal ideal generated by $YX - qXY$. This ring is called the ring of quantum polynomials. Write $x = X + I_q$ and $y = Y + I_q$ and note that $yx = qxy$. It is well known that A_q is a noncommutative Ore domain. Let D_q be its division ring of fractions.

It is enough to describe all signatures of A_q, since there is a bijective correspondence between signatures of A_q and signatures of its division ring of fractions D_q. For every semigroup S let $\mathrm{Tot}(S)$ represent the set of all total orderings of S.

Theorem 12: *There is a bijective correspondence between* $\mathrm{Tot}(\mathbb{Z}\times\mathbb{Z}) \times (\mathbb{Z}\times\mathbb{Z})^*$ *and the set of all signatures of the domain* A_q.

In order to describe total orderings of $\mathbb{Z}\times\mathbb{Z}$ we proceed as follows. Choose a line $\ell \subseteq \mathbb{R}\times\mathbb{R}$ through the origin. This line splits $\mathbb{Z}\times\mathbb{Z}$ in two half–planes. Now declare all elements on one of these half–planes to be positive and the elements from the second half–plane to be negative. If $\ell \cap (\mathbb{Z}\times\mathbb{Z}) \neq \{(0,0)\}$, i.e. the slope of ℓ is a rational number, then one chooses a sign for all elements from $\ell \cap (\mathbb{Z}_{>0}\times\mathbb{Z})$ and gives the elements from $\ell \cap (\mathbb{Z}_{<0}\times\mathbb{Z})$ the opposite sign. This procedure gives all total orderings of $\mathbb{Z}\times\mathbb{Z}$. Note that $(\mathbb{Z}\times\mathbb{Z})^* \cong \mu\times\mu$.

As we are interested in orderings of higher level and thus signatures, it suffices to look at valuations with residue division ring $\mathbb{R}$. We recall a result from Marshall & Zhang [MZ, Theorem 6.2]:

Theorem 13: *There is a bijective correspondence between* $\mathrm{Tot}(\mathbb{Z}\times\mathbb{Z})$ *and the set of all valuation rings of* D_q *with residue division ring* $\mathbb{R}$.

Proposition 14: *A valuation on* D_q *with residue division ring* $\mathbb{R}$ *is real and hence* n*–real for each* n.

Proof. The result follows from the characterization of such valuations on D_q in [MZ, p. 205–206]. ∎

Proof of Theorem 12. Let $n \in \mathbb{N}$. Using the procedure of Corollary 11 we can determine all signatures of level n of D_q and hence of A_q. For every signature χ of D_q there is a compatible valuation ring $A(\chi)$, so that the pushdown of χ is a signature of an archimedean ordering of k_v. By Hölder's theorem, $\mathrm{k}_v \hookrightarrow \mathbb{R}$. On the other hand, $\mathbb{R} \subseteq \mathrm{k}_v$. Since the identity is the only endomorphism of $\mathbb{R}$, we get $\mathrm{k}_v = \mathbb{R}$. As all such valuations are real, they are also n–real. By Theorem 13 there is a bijective correspondence between the set of all such

n–real valuation rings of D_q and $\mathrm{Tot}(\mathbb{Z}\times\mathbb{Z})$. Fix such a n–real valuation v. By [MZ, p.205–206] the valuation group of v is $\mathbb{Z}\times\mathbb{Z}$, so Theorem 10 gives a bijection between all signatures of level n, compatible with v and $(\mathbb{Z}/2n\mathbb{Z}\times\mathbb{Z}/2n\mathbb{Z})^*$. It follows that there is a bijection between $\mathrm{Sgn}^n(D_q)$ and $\mathrm{Tot}(\mathbb{Z}\times\mathbb{Z})\times(\mathbb{Z}/2n\mathbb{Z}\times\mathbb{Z}/2n\mathbb{Z})^*$. We conclude that there is a bijection between $\mathrm{Sgn}(D_q)$ and $\mathrm{Tot}(\mathbb{Z}\times\mathbb{Z})\times(2\mathbb{Z}\times 2\mathbb{Z})^*$ and thus between $\mathrm{Sgn}(D_q)$ and $\mathrm{Tot}(\mathbb{Z}\times\mathbb{Z})\times(\mathbb{Z}\times\mathbb{Z})^*$. This concludes the proof. ■

In the rest of this section we apply the developed valuation theory to study sums of permuted n–th powers in division rings.

Theorem 15: *Let D be a division ring and let $n=2^m\ell$ for $m,\ell\in\mathbb{N}$, where ℓ is odd. Then $a\in\sum D^n$ iff $a\in\sum D^{2^m}$ and $v(a)\in\prod_n\Gamma_v$ for every $\frac{n}{2}$–real valuation v.*

Proof. Let v be a $\frac{n}{2}$–real valuation and $a\in\sum D^n$. Obviously $a\in\sum D^{2^m}$ and $a=p_1+\cdots+p_t$, where $p_i\in\prod_n D^\times$. Without loss of generality $v(p_1)=\max\{v(p_i)\}$. Consequently $a=p_1(1+p_1^{-1}p_2+\cdots+p_1^{-1}p_t)$. Let us denote $q_i=p_1^{-1}p_i$. Clearly $v(q_i)\leqslant 1$. By definition k_v admits an ordering $\overline{P}$ of level 1 such that $\overline{\mathcal{O}_v^\times\cap\prod_n D^\times}\subseteq\overline{P}$. Hence, for the indices i, where $v(q_i)=1$, we have $\overline{q_i}\in\overline{P}^\times$. Therefore $\overline{1+q_2+\cdots+q_t}\in\overline{P}^\times$, so $\overline{1+q_2+\cdots+q_t}\neq 0$ and consequently $1+q_2+\cdots+q_t\in\mathcal{O}_v^\times$. It follows that $v(a)=v(p_1)\in\prod_n\Gamma_v$.

Now let $a\in\sum D^{2^m}$ with $v(a)\in\prod_n\Gamma_v$ for every $\frac{n}{2}$–real valuation v. In order to prove $a\in\sum D^n$ it is enough to show that $a\in P$ for every complete preordering P of exponent n. Let P be a complete preordering of exponent n. There exists a valuation v with $\mathcal{O}_v=A(P)$ such that v is compatible with P. Since the induced ordering $\overline{P}$ of k_v has the property $\overline{\mathcal{O}_v^\times\cap\prod_n D^\times}\subseteq\overline{P}$, v is $\frac{n}{2}$–real. Hence, $v(a)\in\prod_n\Gamma_v$ and there exist $p\in\prod_n D^\times$ and $\varepsilon\in\mathcal{O}_v^\times$ such that $a=p\varepsilon$. As $a,p\in\sum D^{2^m}$, we have $\varepsilon\in\sum D^{2^m}$. It is enough to prove that $\varepsilon\in P$, since this implies $a=p\varepsilon\in P\cdot P\subseteq P$. Suppose that $\varepsilon\notin P$. Since $\mathcal{O}_v^\times\subseteq P\cup -P$, $\varepsilon\in -P$. By Cimprič & Velušček [CV, Corollary 4.3] it follows that $\varepsilon^\ell\in\sum D^n\subseteq P$ and since ℓ is odd, also $\varepsilon^\ell\in -P$. Therefore $\varepsilon^\ell\in P\cap -P=\{0\}$, a contradiction. ■

Lemma 16: *Let $n=2^k\ell$, $k,\ell\in\mathbb{N}$ with ℓ odd. A valuation v is $\frac{n}{2}$–real iff v is 2^{k-1}–real.*

Proof. Let v be 2^{k-1}–real. Then k_v admits an ordering $\overline{P}$ of level 1 such that $\overline{\mathcal{O}_v^\times\cap\prod_{2^k}D^\times}\subseteq\overline{P}$. Since $\prod_n D^\times\subseteq\prod_{2^k}D^\times$, we have $\overline{\mathcal{O}_v^\times\cap\prod_n D^\times}\subseteq\overline{P}$, that is v is also $\frac{n}{2}$–real.

Let v be $\frac{n}{2}$–real. Then k_v admits an ordering $\overline{P}$ of level 1, such that $\overline{\mathcal{O}_v^\times\cap\prod_n D^\times}\subseteq\overline{P}$. It is enough to prove that $\overline{\mathcal{O}_v^\times\cap\prod_{2^k}D^\times}\subseteq\overline{P}$. Let $p\in\prod_{2^k}D^\times$ satisfy $v(p)=1$. Suppose $\overline{p}\notin\overline{P}$. Since $\overline{P}$ is of level 1, we have $\overline{p}\in-\overline{P}$. But by [CV, Corollary 4.3], $p^\ell\in\prod_{2^k\ell}D^\times=\prod_n D^\times$, therefore $\overline{p^\ell}=\overline{p}^\ell\in\overline{\mathcal{O}_v^\times\cap\prod_n D^\times}\subseteq\overline{P}$. On the other hand $\overline{p}^\ell\in(-\overline{P})^\ell=-\overline{P}$, a contradiction. ■

Using this lemma we can generalize Theorem 15:

Corollary 17: *Let D be a division ring and $n = 2^m\ell$, where $m, \ell \in \mathbb{N}$ and ℓ is odd. Then $a \in \sum D^n$ iff $a \in \sum D^{2^m}$ and $v(a) \in \prod_n \Gamma_v$ for every 2^{m-1}–real valuation v.*

This also gives a new proof of a result due to Cimprič, e.g. [Ci2, Theorem 8]. It is the best possible extension of Theorem 1 to the noncommutative setting.

Corollary 18: *For a division ring D and $n \in \mathbb{N}$ the following are equivalent:*

(i) $-1 \notin \sum D^n$,
(ii) $-1 \notin \sum D^{kn}$ *for some odd* $k \in \mathbb{N}$,
(iii) $-1 \notin \sum D^{kn}$ *for all odd* $k \in \mathbb{N}$.

4. Examples revisited

In this section we use the procedure described in Corollary 11 to classify all signatures of level 2ℓ of D_ℓ. For simplicity, we restrict to $k = \mathbb{R}$. Thus $D_\ell = \mathbb{R}(X)(Y;\sigma_\ell)$, where σ_ℓ is the endomorphism of $\mathbb{R}(X)$ fixing $\mathbb{R}$ pointwise and mapping $X \mapsto -X^{2\ell+1}$.

Lemma 19: *The group $G_\ell := \langle a, b \mid ba = a^{2\ell+1}b\rangle$ is orderable. More precisely, there are exactly 4 orderings of G_ℓ, induced by $e < a < b$, $a < e < b$, $b < e < a$ and $b < a < e$.*

Proof. Note that every element of G_ℓ has a unique canonical form $a^i b^j$ for some $i \in \bigcup_{n\in\mathbb{N}} \mathbb{Z}/(2\ell+1)^n =: A$ and $j \in \mathbb{Z}$. Define $N := \{a^\alpha \mid \alpha \in A\}$. Clearly, $N \lhd G$ and $G/N = \langle bN\rangle \cong \mathbb{Z}$. As

$$(a^\alpha b^n)\left(a^\beta b^q\right)(a^\alpha b^n)^{-1} = a^{\alpha+(2\ell+1)^n\cdot\beta-(2\ell+1)^q\cdot\alpha}b^q,$$

two elements of the form $a^\alpha b$ are always conjugate. So, for every total ordering $\leqslant$ of G_ℓ we have $\forall\alpha \in A:\ a^\alpha > b$ or $\forall\alpha \in A:\ a^\alpha < b$. This shows that N is convex in every ordering of G_ℓ. Hence there are at most 4 different orderings of G_ℓ, induced by $e < a < b$, $a < e < b$, $b < e < a$ and $b < a < e$. To finish the proof, note that

$$\begin{aligned}
a^{\alpha_1}b^{n_1} \succ_1 a^{\alpha_2}b^{n_2} \quad &:\Longleftrightarrow \quad n_1 > n_2 \vee (n_1 = n_2 \wedge \alpha_1 > \alpha_2),\\
a^{\alpha_1}b^{n_1} \succ_2 a^{\alpha_2}b^{n_2} \quad &:\Longleftrightarrow \quad n_1 > n_2 \vee (n_1 = n_2 \wedge \alpha_1 < \alpha_2),\\
a^{\alpha_1}b^{n_1} \succ_3 a^{\alpha_2}b^{n_2} \quad &:\Longleftrightarrow \quad n_1 < n_2 \vee (n_1 = n_2 \wedge \alpha_1 > \alpha_2),\\
a^{\alpha_1}b^{n_1} \succ_4 a^{\alpha_2}b^{n_2} \quad &:\Longleftrightarrow \quad n_1 < n_2 \vee (n_1 = n_2 \wedge \alpha_1 < \alpha_2)
\end{aligned}$$

are explicit descriptions of these orderings as a short calculation shows. ∎

Fix one of the total orderings of G_ℓ constructed in this proof. Define

$$\begin{aligned}
v_\ell : R_\ell \setminus \{0\} \quad &\longrightarrow \quad G_\ell,\\
\sum c_{m,n}X^mY^n \quad &\longmapsto \quad \max\{a^{m_k}b^{n_k} \mid c_{m_k,n_k} \neq 0\}.
\end{aligned}$$

Lemma 20: *v_ℓ is a valuation on R_ℓ.*

Proof. Let $0 \neq f = \sum a_{m,n} X^m Y^n \in R_\ell$ and $0 \neq g = \sum b_{i,j} X^i Y^j \in R_\ell$. Assume $v(f) = a^{m_0} b^{n_0}$ and $v(g) = a^{i_0} b^{j_0}$. If $a^{m_0} b^{n_0} \neq a^{i_0} b^{j_0}$, then $v(f+g) = \max\{a^{m_0} b^{n_0}, a^{i_0} b^{j_0}\}$. If $a^{m_0} b^{n_0} = a^{i_0} b^{j_0}$ and $a_{m_0,n_0} \neq -b_{i_0,j_0}$, then $v(f+g) = \max\{v(f), v(g)\}$. Otherwise the highest term of $f+g$ is smaller than the highest term of f and g and thus $v(f+g) < \max\{v(f), v(g)\}$. It is clear that $v(fg) = v(f)v(g)$. ∎

The following result is classical. Therefore its proof is omitted.

Proposition 21: *Let R be a left Ore domain with division ring of left fractions D. If $v : R \longrightarrow \Gamma \cup \{0\}$ is a valuation with value group Γ, then there is a unique extension of v to a valuation $v : D \longrightarrow \Gamma \cup \{0\}$. The extension is defined by $v\big(ab^{-1}\big) := v(a)v(b)^{-1}$.*

Now extend the valuation v_ℓ to a valuation on D_ℓ.

Theorem 22: *The valuation $v_\ell : D_\ell \longrightarrow G_\ell \cup \{0\}$ is 2ℓ–real, but not ℓ–real. Moreover, $\mathrm{k}_v = \mathbb{R}$.*

Proof. There are 4 cases to distinguish. As all proofs are the same, w.l.o.g. we assume that the ordering of G_ℓ is given by the relation $e < a < b$.

By the valuation criterion developed in Section 3 and Proposition 2, v_ℓ cannot be ℓ–real. To show that it is 2ℓ–real, it suffices to show that v_ℓ is compatible with an ordering of level 2ℓ, which induces an ordering of level 1 of k_v. For this, let σ be the signature defined in Proposition 3. We will show that $1 + \mathfrak{m}_v \subseteq P_\sigma$. If $x = fg^{-1} \in \mathfrak{m}_v$, then $1 + x = (f+g)g^{-1}$. Thus $\sigma(x) = \sigma(f+g)\sigma(g)^{-1}$. From the definition of σ and the fact $x \in \mathfrak{m}_v$, it follows that $\sigma(x) = 1$.

To show that $\mathrm{k}_v = \mathbb{R}$, we define a mapping

$$\begin{aligned} \pi : \mathcal{O}_v &\longrightarrow \mathbb{R}, \\ fg^{-1} = \frac{a_{mn} X^m Y^n + (\text{lower terms})}{b_{ij} X^i Y^j + (\text{lower terms})} &\longmapsto \begin{cases} a_{mn} b_{ij}^{-1} & \mid fg^{-1} \in \mathcal{O}_v \setminus \mathfrak{m}_v \\ 0 & \mid \text{otherwise.} \end{cases} \end{aligned}$$

It is clear that π is a surjective ring homomorphism with kernel $\mathfrak{m}_v$. Hence $\mathrm{k}_v = \mathbb{R}$, and consequently the ordering, induced by σ, has level 1. ∎

Proposition 23: *Let v be a 2ℓ–real valuation on D_ℓ. Then:*

(1) *$v(X)$ and $v(Y)$ are independent.*
(2) *The value group Γ_v of v is isomorphic to G_ℓ.*

Proof. (1)[1] Suppose otherwise. Then $v(X^rY^s) = 1$ with $r, s \in \mathbb{Z}$ not both zero. Applying v to each side of $YX^r = (-1)^r X^{(2\ell+1)r}Y$ yields $v(Y)v(Y^{-s}) = v(Y^{-(2\ell+1)s})v(Y)$, i.e., $v(Y^{2\ell s}) = 1$ so $v(Y^s) = 1$ and consequently also $v(X^r) = 1$. If $r \neq 0$, then $v(X) = 1$ and the equation $YXY^{-1}X^{-1} = -X^{2\ell}$ shows that v is not 2ℓ–real. If $r = 0$ then $s \neq 0$ so $v(Y) = 1$ and this same equation shows that $v(X^{2\ell}) = 1$, i.e., $v(X) = 1$. But we have already seen that this is impossible.

(2) Since $v(X)$ and $v(Y)$ are independent, this determines the valuation v on R_ℓ and consequently on D_ℓ by Proposition 21. Since $v(Y)v(X) = v(X)^{2\ell+1}v(Y)$, the value group Γ_v is an ordered quotient group of G_ℓ, say $\Gamma_v = G_\ell/N$. We claim that $N = \{1\}$. Obviously, $N \subsetneq G_\ell$. Let $a^\alpha b^q \in N$ for $\alpha \in A$ and $q \in \mathbb{Z}$ (see the proof of Lemma 19). As in Lemma 19 we can deduce that $N \supseteq \{a^\beta b^q \mid \beta \in A\} =: N_1$. If $q = 0$, then $N_1 = \{a^\beta \mid \beta \in A\}$ and hence $G_\ell/N_1 = \mathbb{Z}$. Since G_ℓ/N is ordered, we have $N_1 = N$ and thus $\Gamma_v = \mathbb{Z}$. But then $v(X)$ and $v(Y)$ cannot be independent. This contradiction shows that $q \neq 0$. Hence $b^q \in N_1$. Since G_ℓ/N is ordered and hence torsion–free, this implies $b \in N$. But then $G_\ell = N$, a contradiction. Thus $N = \{1\}$ and $G_\ell = \Gamma_v$. ■

This proposition shows that every 2ℓ–real valuation (with archimedean residue division ring) on D_ℓ is of the form constructed above. Furthermore, given such a valuation v, we can choose a signature η of D_ℓ of level 2ℓ that is compatible with v as in Proposition 3. By Corollary 11, all other such signatures are obtained as $\chi = \eta \cdot (\tau \circ v)$, where $\tau \in \big(G_\ell/\prod_{4\ell} G_\ell\big)^*$. To finish the construction, observe that a character τ of $G_\ell/\prod_{4\ell} G_\ell$ is given by $\xi_1 := \tau\big(a\prod_{4\ell} G_\ell\big) \in \mathbb{C}$ and $\xi_2 := \tau\big(b\prod_{4\ell} G_\ell\big) \in \mathbb{C}$ satisfying $\xi_1^{2\ell} = 1$ and $\xi_2^{4\ell} = 1$.

5. Real places and a question of Marshall & Zhang

In [MZ] Marshall & Zhang defined real places and order compatible real places on domains. They showed that an order compatible real place "comes" from an ordering of level 1, see [MZ, Theorem 2.5]. They asked whether a real place is necessarily order compatible (see the paragraph before Theorem 2.5 in [MZ]). We use the examples constructed in Section 2 to show this is not always the case.

For the convenience of the reader, we recall the definition of real places.

Definition: A *real place* α on a domain A is a map $(A \times A) \setminus \{(0,0)\} \longrightarrow \mathbb{R} \cup \{\infty\}$ satisfying

(i) $\alpha(a,b) = \infty \;\Leftrightarrow\; \alpha(b,a) = 0$.
(ii) $\alpha(a,b) \neq \infty \;\wedge\; \alpha(b,c) \neq \infty \;\Rightarrow\; \alpha(a,b)\alpha(b,c) = \alpha(a,c)$.
(iii) $\alpha(a,c) \neq \infty \;\wedge\; \alpha(b,c) \neq \infty \;\Rightarrow\; \alpha(a,c) + \alpha(b,c) = \alpha(a+b,c)$.
(iv) $\alpha(a,b) = \alpha(ac,bc) = \alpha(ca,cb)$ for any $0 \neq c \in A$.

[1] This argument was provided to us by the referee.

In case D is a division ring, the definition of a real place on D can be somewhat simplified, by viewing elements $(a,b) \in D \times D^{\times}$ as fractions $a/b \in D$. In this case, a real place on D is a mapping $\alpha : D \longrightarrow \mathbb{R} \cup \{\infty\}$ satisfying

(i) $\alpha(a) = \infty \;\Leftrightarrow\; \alpha\big(a^{-1}\big) = 0$.
(ii) $\alpha(a) \neq \infty \;\wedge\; \alpha(b) \neq \infty \;\Rightarrow\; \alpha(a)\alpha(b) = \alpha(ab)$.
(iii) $\alpha(a) \neq \infty \;\wedge\; \alpha(b) \neq \infty \;\Rightarrow\; \alpha(a) + \alpha(b) = \alpha(a+b)$.
(iv) $\alpha\big(cac^{-1}\big) = \alpha(a)$ for any $0 \neq c \in D$.

Assume P is an ordering of level n of D. As explained in Section 3, there is a natural valuation ring $A(P)$ with maximal ideal $I(P)$ associated to P. Moreover, P induces an archimedean ordering of $A(P)/I(P)$. In particular, there is an order preserving embedding $A(P)/I(P) \hookrightarrow \mathbb{R}$. Hence we have a mapping $\alpha : A(P) \longrightarrow \mathbb{R}$. We extend this to a mapping $\alpha : D \longrightarrow \mathbb{R} \cup \{\infty\}$ by setting $\alpha\big(D \setminus A(P)\big) = \{\infty\}$. It is an easy calculation to show that $\alpha : D \longrightarrow \mathbb{R} \cup \{\infty\}$ is a real place. This real place is *associated to* P and we write $\alpha_P := \alpha$.

Definition: A real place of the form α_P for an ordering of level n is called *n–order compatible.*

As explained in [MZ] in the paragraph after Proposition 2.4, a real place α on D gives rise to a valuation $v_\alpha : D \longrightarrow \Gamma_\alpha \cup \{0\}$. Equivalently, one can observe that $A_\alpha := \alpha^{-1}(\mathbb{R})$ is an (invariant) valuation ring with maximal ideal $I_\alpha := \alpha^{-1}(0)$.

The following theorem resembles an analogous result for order compatible real places, see [MZ, Theorem 2.5]. Our proof is a generalization of the one given there – merely one implication requires more work.

Definition: Let $n \in \mathbb{N}$ and let D be a division ring. For $S \subseteq D$ write $D^{2n}(S)$ for the set of all permuted products of $2n$–th powers of elements of D and elements of S.

Theorem 24: *Let α be a real place on D and define $S_\alpha := \{x \in D \mid \alpha(x) \in \mathbb{R}_{>0}\}$. Then the following are equivalent:*

(i) *For $a \in D^{2n}(S_\alpha)$, $\alpha(a) \in [0,\infty]$.*
(ii) *If $a_1,\dots,a_n \in D^{2n}(S_\alpha) \setminus \{0\}$, then $v_\alpha(a_1 + \cdots + a_n) = \max\{v_\alpha(a_i) \mid i = 1,\dots,n\}$.*
(iii) $-1 \notin \sum D^{2n}(S_\alpha)$.
(iv) *There exists a level n ordering P of D containing S_α.*
(v) *α is n–order compatible.*
(vi) $D^{2n}(S_\alpha) \cap -S_\alpha = \varnothing$.

Proof. (i) $\Rightarrow$ (ii): Assume $v_\alpha(a_1) = \max\{v_\alpha(a_i) \mid i = 1,\dots,n\}$. Then $\alpha(a_i/a_1) \geqslant 0$ and $\alpha(a_1/a_1) = \alpha(1) = 1$, so $\alpha(\sum_{i=1}^n a_i/a_1) = \sum_{i=1}^n \alpha(a_i/a_1) > 0$. Hence $v_\alpha(\sum_{i=1}^n a_i/a_1) = 1$.

(ii) $\Rightarrow$ (iii): Otherwise, there are $0 \neq a_1,\dots,a_n \in D^{2n}(S_\alpha)$ with $1 + a_1 + \cdots + a_n = 0$. This clearly contradicts (ii).

(iii) $\Rightarrow$ (iv): This is the classical Artin–Schreier type result for orderings of higher level. See e.g. [Po1, Theorem 3.13].

(iv) $\Rightarrow$ (v): We will show that $\alpha = \alpha_P$. Take an $x \in D$ with $\alpha(x) \neq \infty$. For $r_1, r_2 \in \mathbb{Q}$, $r_1 < \alpha(x) < r_2$ implies $\alpha(x - r_1) > 0$ and $\alpha(r_2 - x) > 0$. By the definition of S_α and (iv), $x - r_1 \in S_\alpha \subseteq P$ and $r_2 - x \in S_\alpha \subseteq P$. Now assume $x \in P$. Then $r_2 \pm x \in P$ and hence $|\alpha_P(x)| \leqslant r_2$. This shows $x \in A(P)$. In particular, $r_2 - x \in P \cap A(P)$ and thus $\alpha_P(r_2 - x) \geqslant 0$. This gives $r_1 \leqslant \alpha_P(x) \leqslant r_2$. If $x \notin P$, we can write

$$x = \frac{1}{(2n)!} \sum_{d=0}^{2n-1} (-1)^{2n-1-d} \binom{2n-1}{d} \big((x+d)^{2n} - d^{2n}\big).$$

Now by properties of real places, $\alpha(x) = \alpha_P(x)$. If $\alpha(x) = \infty$, then $\alpha\big(\frac{1}{x}\big) = 0$ and by the above, $\alpha_P\big(\frac{1}{x}\big) = 0$. Hence $\alpha_P(x) = \infty$.

(v) $\Rightarrow$ (vi): As $S_\alpha \subseteq P$, $D^{2n}(S_\alpha) \cap -S_\alpha \subseteq P \cap -P = \{0\}$. But $0 \notin S_\alpha$.

(vi) $\Rightarrow$ (i): Otherwise, there is some $a \in D^{2n}(S_\alpha)$ satisfying $\alpha(a) < 0$. Then $-a \in S_\alpha$, so $a \in D^{2n}(S_\alpha) \cap -S_\alpha = \varnothing$. ∎

Example: Finally, we are able to give an n–order compatible real place that is not order compatible, thereby answering the question of Marshall & Zhang. Take the division ring D_1 constructed in Section 2 and let P be an ordering of level 2 of D_1. As explained above, $\alpha_P : D_1 \longrightarrow \mathbb{R} \cup \{\infty\}$ is a 2–order compatible real place. But by Theorems 24 and 4, α_P cannot be order compatible.

This example raises another question. Is every real place n–order compatible for some n? Our final example settles this.

Example: Let $K := \mathbb{Q}(\sqrt{2})$ and let $\omega : K \longrightarrow K$ be the automorphism sending $\sqrt{2} \longmapsto -\sqrt{2}$. We form $F := K(\!(\mathbb{Z}, \omega)\!)$ the division ring of skew Laurent series. Elements of F are series of the form $\sum_{k=n}^{\infty} r_k x^k$ for $r_k \in K$ and $n \in \mathbb{Z}$.

To show that F has no orderings of any level, assume $\sigma : F \longrightarrow \mathbb{C}$ is a signature. Then $\sigma(\sqrt{2}x) = \sigma(\sqrt{2})\sigma(x) = \sigma(x)\sigma(\sqrt{2}) = \sigma(x\sqrt{2}) = \sigma(-\sqrt{2}x) = -\sigma(\sqrt{2}x)$, a contradiction.

We form a mapping $v : F \longrightarrow \mathbb{Z} \cup \{\varnothing\}$, where $\varnothing$ is considered to be the smallest element in $\mathbb{Z} \cup \{\varnothing\}$, by sending $0 \longmapsto \varnothing$ and $v\big(\sum_{k=n}^{\infty} r_k x^k\big) := -n$, if $r_n \neq 0$. As a short calculation shows, v is a valuation. Moreover, the corresponding valuation ring is $\mathcal{O}_v = \big\{ \sum_{k=n}^{\infty} r_k x^k \mid n \in \mathbb{N}_0 \big\}$, while its maximal ideal is $\mathfrak{m}_v = \big\{ \sum_{k=n}^{\infty} r_k x^k \mid n \in \mathbb{N} \big\}$. The residue division ring is $\mathcal{O}_v / \mathfrak{m}_v = K$ and is formally real, although F has no orderings of any level.

By fixing an embedding $K \hookrightarrow \mathbb{R}$, we get a map $\mathcal{O}_v \longrightarrow \mathbb{R}$. This can be extended to $\alpha : D \longrightarrow \mathbb{R} \cup \{\infty\}$ by $\alpha(x) := \infty$ if $x \in D \setminus \mathcal{O}_v$. Now α is a real place that is clearly not n–order compatible for any n. This example also gives an almost real valuation that is not n–real for any n.

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