Comparison theorems for o-minimal singular (co)homology

Mário J. Edmundo *
CMAF Universidade de Lisboa
Av. Prof. Gama Pinto 2
1649-003 Lisboa, Portugal
and
Arthur Woerheide
6203 S. Evans Ave.
Chicago, IL 60637
U.S.A

April 10, 2004

Abstract

We show here the existence of the o-minimal simplicial and singular (co)homology in o-minimal expansions of real closed fields and prove several comparison theorems for o-minimal (co)homology theories.

*Supported by the FCT grant SFRH/BPD/6015/2001 and partially by the European Research and Training Network HPRN-CT-2001-00271 RAAG. *MSC:* 03C64; 55N10. *Keywords and phrases:* O-minimal singular homology and cohomology. Revised on September 4, 2006.

1 Introduction

As it is known o-minimal geometry is a wide ranging generalization of semi-algebraic, sub-analytic and sub-pfaffian geometry. In this paper we will work over an o-minimal expansion $\mathcal{N}$ of an ordered ring (necessarily a real closed field) and definable means definable in $\mathcal{N}$ possibly with parameters.

Associated to the first-order theory $\mathrm{Th}(\mathcal{N})$ of $\mathcal{N}$ over N we have the categories of definable sets and of definably compact definable sets (or equivalently of closed and bounded definable sets, see [ps]). The morphisms on these categories are continuous definable maps. For basic o-minimality and the notion of definable sets and definable maps we refer the reader to [vdd].

By [KPS] and the first-order logic compactness theorem, for each infinite cardinal κ, the theory $\mathrm{Th}(\mathcal{N})$ has a model of cardinality κ and any such model is also an o-minimal structure. Moreover, by [Sh], for every $\kappa > \max\{\aleph_0, |\mathrm{Th}(\mathcal{N})|\}$, there are up to isomorphism 2^κ models $\mathcal{M}$ of $\mathrm{Th}(\mathcal{N})$ such that $|M| = \kappa$. A model $\mathcal{M}$ of $\mathrm{Th}(\mathcal{N})$ determines the categories of definable sets and of definably compact definable sets defined over M. Since we are assuming that $\mathrm{Th}(\mathcal{N})$ is the theory of $\mathcal{N}$ over N, the model $\mathcal{M}$ detemines a functor from the category of definable sets (resp., of definably compact definable sets) (defined over N) into the category of definable sets (resp., of definably compact definable sets) defined over M. In o-minimal geometry we are interested in developing a geometry which is invariant under this functor.

For example, if $\mathcal{N}$ is the field $\mathbb{R}$ of real numbers (resp., the expansion $\mathbb{R}_{\mathrm{an}}$ of $\mathbb{R}$ by all globally analytic functions), then the geometry of the category of definable sets is semi-algebraic geometry (resp., global sub-analytic geometry). While the geometry on models of $\mathbb{R}$, which are exactly the real closed fields, is already well developed within semi-algebraic geometry (see [BCR]), the geometry in arbitrary models of $\mathbb{R}_{\mathrm{an}}$ and of other o-minimal structures on the real numbers such as $\mathbb{R}_{\mathrm{exp}}, \mathbb{R}_{\mathrm{an,\,exp}}, \mathbb{R}_{\mathrm{an}^*}, \mathbb{R}_{\mathrm{an}^*,\,\mathrm{exp}}$, etc. is still being developed, and the present paper is a contribution in this direction. For the o-minimality of the structures $\mathbb{R}_{\mathrm{an}}$, $\mathbb{R}_{\mathrm{exp}}, \mathbb{R}_{\mathrm{an,\,exp}}, \mathbb{R}_{\mathrm{an}^*}, \mathbb{R}_{\mathrm{an}^*,\,\mathrm{exp}}$ see resp., [dd], [w], [dm], [ds1] and [ds2].

We should also point out that we will be working in a bigger generality since we will not assume that the first order theory $\mathrm{Th}(\mathcal{N})$ of $\mathcal{N}$ has a model with the order type of the reals.

An important tool for developing the geometry of definable sets would be of course a homology and cohomology theory for this category. Such

(co)homology theories are defined by the Eilenberg-Steenrod (co)homology axioms adapted to the o-minimal setting (see [Wo] or Sections 3 and 6 here).

In the thesis [Wo], the second author constructs the o-minimal simplicial homology for the category of definably compact sets and the o-minimal singular homology for the category of definable sets. From this one easily constructs the o-minimal simplicial cohomology and the o-minimal singular cohomology (see [e] or Section 6 here). Note that since the proof of the existence of the classical topological simplicial (resp., singular) homology theory uses local compactness of closed simplicial complexes, namely for proving the simplicial approximation theorem (resp., the local compactness of the standard simplices, namely in the proof of the excision axiom) the classical techniques do no generalize to the o-minimal case.

Since the thesis [Wo] has not been published, the construction of the o-minimal singular and simplicial homology theory will be included here in Sections 4 and 5.

As observed in [Wo], the definable triangulation theorem ([vdd]) and a purely algebraic argument, imply as in the classical topological case in [dp] Chapter 6 (6.3) the following (where here and below, Q is a module over a commutative ring R with unit element):

Theorem 1.1 *Any two o-minimal (co)homology theories in the category of definably compact definable sets with coefficients in Q are isomorphic. In particular, any o-minimal (co)homology theory in the category of definably compact definable sets with coefficients in Q is isomorphic to the o-minimal simplicial (co)homology with coefficients in Q.*

Note that Theorem 1.1 is about definably compact definable sets which are analogues of the semi-algebraically complete sets from real algebraic geometry and are not in general compact sets. Thus this is not a consequence of the classical uniqueness theorem for homology theories on the category of compact topological triangulable spaces given in [es] Chapter III, Theorem 10.1.

One of the main results of the paper is the following generalization of Theorem 1.1 to the category of definable sets. Although definable sets are also definably triangulable ([vdd]) this result is not the same as the classical result [es] Chapter III, Theorem 10.1 since there simplicial complexes are compact unlike in the o-minimal case where they are not assumed to be closed.

Theorem 1.2 *Any two o-minimal (co)homology theories on the category of definable sets with coefficients in Q are isomorphic. In particular, any o-minimal (co)homology theory in the category of definable sets with coefficients in Q is isomorphic to the o-minimal singular (co)homology with coefficients in Q.*

This uniqueness theorem implies the natural invariance of the o-minimal singular (co)homology in elementary extensions and in o-minimal expansions of $\mathcal{N}$. Also, in the case that $\mathcal{N}$ is an o-minimal expansion of the field of real numbers, it follows that the o-minimal singular (co)homology is naturally isomorphic to the usual topological singular (co)homology. Here is the homology version of what we mean:

Corollary 1.3 *Suppose that $\mathcal{M}$ is an elementary extension or an o-minimal expansion of $\mathcal{N}$. Let $A \subseteq X$ and $B \subseteq Y$ be definable sets over N and $f : X \longrightarrow Y$ a continuous definable map over N such that $f(A) \subseteq B$. Then we have isomorphisms $i_*^{\mathcal{M}} : H_*(X, A; Q) \longrightarrow H_*(X(M), A(M); Q)$ making the following diagram*

$$\begin{array}{ccc} H_*(X, A; Q) & \xrightarrow{f_*} & H_*(Y, B; Q) \\ \downarrow{\scriptstyle i_*^{\mathcal{M}}} & & \downarrow{\scriptstyle i_*^{\mathcal{M}}} \\ H_*(X(M), A(M); Q) & \xrightarrow{(f^M)_*} & H_*(Y(M), B(M); Q) \end{array}$$

commutative.

It is possible to obtain a variant of Theorem 1.2 that does not assume o-minimality. This is a generalization of the classical result [es] Chapter III, Theorem 10.1 since there simplicial complexes are compact unlike here where they are not assumed to be closed.

Theorem 1.4 *Any two (co)homology theories on the category of finitely triangulable topological spaces over the real numbers with coefficients in Q are isomorphic.*

The natural invariance of the o-minimal simplicial (co)homology in elementary extensions was already pointed out in [bo] and [e] as it follows from the construction of the o-minimal simplicial (co)homology.

The existence of a semi-algebraic (co)homology theory is proved in [D1] (see also [D2] and [dk]), in a rather complicated way, via sheaf cohomology theory with constant coefficients for the semi-algebraic site. Using the semi-algebraic triangulation theorem from [dk], it is proved in [D1] (see also [dk]) that the semi-algebraic (co)homology groups of definably compact definable sets coincide with the simplicial (co)homology groups of the corresponding simplicial complexes. Furthermore, in [D2], the natural invariance of semi-algebraic (co)homology theory in elementary extensions is proved.

In [k], Knebusch states that it would be desirable to have a more elementary proof of the existence of the semi-algebraic singular (co)homology. The thesis [Wo] provides such elementary proofs.

The main theorems of the paper are a combination of homological algebra, o-minimality and pure model theory. From homological algebra we need, among other things, the method of acyclic models and the theorem on removing cells. From o-minimality the main result we need is the definable triangulation theorem, and from model theory we require the reader's familiarity with the notions of models and elementary extensions of models of a theory.

Acknowledgements. The first author would like to thank Gareth Owen Jones who gave an Advanced Logic Class at Oxford University based on previous versions of this paper. His comments and questions were quite useful.

2 Tools from homological algebra

In this section we recall basic notions from homological algebra together with the method of acyclic models and the theorem on removing cells. We will follow here [d] Chapter II and Chapter VI, Section 11 where the reader can find all the details.

Below R will be a commutative ring with unit, **Comp** denotes the category of chain complexes of R-modules and $\widetilde{\mathbf{C}}\mathbf{omp}$ the category of augmented chain complexes of R-modules i.e. the subcategory whose objects are all those chain complexes $E_* = (E_*, \partial_*)$ with $E_{-1} = \mathbb{Z}$, and $E_n = 0$ for $n < -1$, and whose morphisms are those chain maps f between such chain complexes with $f_{-1} = 1_{\mathbb{Z}}$.

We say that a chain complex $E_* = (E_*, \partial_*)$ is *acyclic* if $H_n(E_*) = \mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. The chain complex E_* is said to be *free* if E_n is a free R-module for all $n \in \mathbb{Z}$. A family $U_* = (U_n)_{n\in\mathbb{Z}}$ is a *basis of* E_* if U_n is a basis for E_n for each $n \in \mathbb{Z}$; if U_* is a basis for E_*, $s \in U_n$ and $t \in E_n$, we say that s is *involved in* t if, when t is written as an R-linear combination of elements of U_n, s has a non zero coefficient.

A subcomplex $F_* = (F_*, \partial_*)$ is an *adequate subcomplex* of E_* if (i) whenever $z \in \mathrm{Ker}\partial_n$, then there is $z' \in \mathrm{Ker}\partial_n \cap F_n$ such that $z - z' \in \mathrm{Im}\partial_{n+1}$ and (ii) if $z' \in \mathrm{Ker}\partial_n \cap F_n$ and $z' = \partial_{n+1}c$ for some $c \in E_{n+1}$, then there is $c' \in F_{n+1}$ with $z' = \partial_{n+1}c'$.

Note that if F_* is an adequate subcomplex of E_*, then the inclusion $i : F_* \longrightarrow E_*$ induces isomorphisms $H_n(i) : H_n(F_*) \longrightarrow H_n(E_*)$ for all $n \in \mathbb{Z}$. Later we will need the following result from [Wo].

Theorem 2.1 (Removing cells) *Let E_* be a chain complex with basis U_*, $s \in U_n$ and $t \in U_{n+1}$ such that $s = \partial_{n+1}t + r$ with s not involved in r and such that s is not involved in $\partial_{n+1}t'$ for all $t' \in U_{n+1} - \{t\}$. Let $F_m = E_m$ for $m \neq n, n+1$, $F_n = \langle U_n - \{s\}\rangle$ and $F_{n+1} = \langle U_{n+1} - \{t\}\rangle$. Then F_* with the restricted boundary maps forms an adequate subcomplex of E_*.*

Proof. It is easy to see that F_* is indeed a subcomplex of E_*, i.e. $\partial_*(F_m) \subseteq F_{m-1}$ for all $m \in \mathbb{Z}$. The case $m = n+2$ is the only case that needs to be checked. But if r is a $(n+2)$-cell such that t is involved in $\partial_{n+2}r$, then s is involved in $\partial_{n+1} \circ \partial_{n+2}r$, so that $\partial_{n+1} \circ \partial_{n+2}r \neq 0$, which is a contradiction.

It remains to verify conditions (i) and (ii) of the definition of an adequate subcomplex. This is trivial for all dimensions other than $n+1, n$ and $n-1$. In dimension $n+1$, condition (i) follows from the fact that if $z \in \mathrm{Ker}\partial_n$, then z does not involve t and so $z \in \mathrm{Ker}\partial_n \cap F_n$; on the other hand, condition (ii) follows from the fact that $F_{n+2} = E_{n+2}$.

We now consider the case of dimension n. We start with condition (i). Let $z \in \mathrm{Ker}\partial_n$. Write $z = ms + \beta$, with s not involved in β. Let $z' = z - m\partial_{n+1}t = \beta + mr$. Then $\partial_n z' = \partial_n z - m\partial_n \circ \partial_{n+1}t = 0$, so $z' \in \mathrm{Ker}\partial_n \cap F_n$. Furthermore, $z - z' = m\partial_{n+1}t = \partial_{n+1}(mt) \in \mathrm{Im}\partial_{n+1}$. As for condition (ii), suppose that $z' \in \mathrm{Ker}\partial_n \cap F_n$ and $z' = \partial_{n+1}c$ for some $c \in E_{n+1}$. Write $c = mt + c'$, with t not involved in c'. Then $\partial_{n+1}c = ms - mr + \partial_{n+1}c'$. Since $\partial_{n+1}c = z' \in \mathrm{Ker}\partial_n \cap F_n$, $\partial_{n+1}c$ does not involve s. Therefore, $m = 0$ and $c \in F_{n+1}$.

In dimension $n-1$, condition (i) follows from the fact that $\mathrm{Ker}\partial_{n-1} \cap F_{n-1} = \mathrm{Ker}\partial_{n-1}$. Suppose now that $z' \in \mathrm{Ker}\partial_{n-1} \cap F_{n-1}$ and $z' = \partial_n c$ for some $c \in E_n$. Write $c = ms + c'$, with $c' \in F_n$. Then $c - m\partial_{n+1}t = c' + mr \in F_n$, and $\partial_n(c' + mr) = \partial_n c - m\partial_n \circ \partial_{n+1}t = \partial_n c = z'$. Thus condition (ii) holds as well. $\square$

Remark 2.2 In Section 4 we will use the special version of Theorem 2.1 where the chain complex E_* is the chain complex of a closed simplicial complex L and F_* is the subchain complex of E_* associated to a closed simplicial subcomplex M of L obtained by removing two appropriate simplexes.

A chain map $f : E_* \longrightarrow F_*$ is a *chain equivalence* if there is a chain map $g : F_* \longrightarrow E_*$ such that $g \circ f \simeq 1_{E_*}$ and $f \circ g \simeq 1_{F_*}$. Here, $f \simeq h$ means that the chain maps $f : E_* \longrightarrow F_*$ and $h : E_* \longrightarrow F_*$ are *chain homotopic*, i.e. there is a sequence of maps $s_n : E_n \longrightarrow F_{n+1}$ such that $f_n - h_n = \delta_{n+1} \circ s_n + s_{n-1} \circ \partial_n$ for all $n \in \mathbb{Z}$. This is an equivalence relation compatible with composition. **hComp** will denote the category whose objects are chain complexes and whose morphisms are homotopy classes of chain maps. Note that we have a natural functor **Comp** $\longrightarrow$ **hComp**. Similarly, we define $\widetilde{\mathbf{hComp}}$ and the functor $\widetilde{\mathbf{Comp}} \longrightarrow \widetilde{\mathbf{hComp}}$.

The following will be used quite often in this paper. If $f \simeq g$, then $H_n(f) = H_n(g)$ for every $n \in \mathbb{Z}$. Also if $f : E_* \longrightarrow F_*$ is a chain equivalence, then $H_n(f) : H_n(E_*) \longrightarrow H_n(F_*)$ is an isomorphism for every $n \in \mathbb{Z}$. Conversely, by [d] Chapter II, Proposition 4.3, if $f : E_* \longrightarrow F_*$ is a chain map between free chain complexes such that $H_n(f) : H_n(E_*) \longrightarrow H_n(F_*)$ is an isomorphism for every $n \in \mathbb{Z}$, then $f : E_* \longrightarrow F_*$ is a chain equivalence.

The method of acyclic models is useful for constructing natural chain maps and natural chain homotopies.

Definition 2.3 Let F be a functor from a category $\mathbf{C}$ into the category of R-modules and let $\mathbf{M}$ be a *set of models* for $\mathbf{C}$ (i.e., a subset of Obj$\mathbf{C}$). We say that F is *free with basis in* $\mathbf{M}$ if FC is a free R-module for every $C \in \mathrm{Obj}\mathbf{C}$ and there are indexed families $(M_j)_{j\in J}$ of models in $\mathbf{M}$ and of elements $(x_j)_{j\in J}$ with $x_j \in FM_j$ such that, for every $C \in \mathrm{Obj}\mathbf{C}$, the set $\{F(\sigma)(x_j) : j \in J, \sigma : M_j \longrightarrow C \in \mathrm{Mor}\mathbf{C}\}$ is a basis of FC. In this situation, we call x a *basis of* F *in* $\mathbf{M}$.

Let $E : \mathbf{C} \longrightarrow \mathbf{Comp}$ be a functor. We say that E is *free with basis in* $\mathbf{M}$ if E_n is free with basis in $\mathbf{M}$ for each $n \in \mathbb{Z}$. We say that E is nonnegative if EC is a nonnegative chain complex for all $C \in \mathrm{Obj}\mathbf{C}$; E is *acyclic on a set of models* $\mathbf{M}$ for $\mathbf{C}$ if EM is an acyclic chain complex for every $M \in \mathbf{M}$. A natural transformation between functors $E, F : \mathbf{C} \longrightarrow \mathbf{Comp}$ is called a natural chain map.

An *augmented natural chain map* $\tau : E \longrightarrow F$ is by definition a natural transformation $\tau : E \longrightarrow F$ between two functors $E, F : \mathbf{C} \longrightarrow \widetilde{\mathbf{C}}\mathbf{omp}$. Note that in this case, for every $C \in \mathrm{Obj}\mathbf{C}$, the map $(\tau_{\mathbf{C}})_{-1} : E_{-1}C \longrightarrow F_{-1}C$ is the identity 1_R. An augmented natural chain map $\tau : E \longrightarrow F$ is a *natural chain equivalence* if, for every $C \in \mathrm{Obj}\mathbf{C}$, the chain map $(\tau_C)_* : E_*C \longrightarrow F_*C$ is a chain equivalence of chain complexes which is natural with respect to the morphisms of $\mathbf{C}$. The notion of two natural chain maps $\tau, \rho : E \longrightarrow F$ being *naturally chain homotopic* is defined in a similar way.

Theorem 2.4 (Acyclic Models in $\widetilde{\mathbf{C}}\mathbf{omp}$) *Let* $\mathbf{C}$ *be a category with set of models* $\mathbf{M}$ *and let* $E, F : \mathbf{C} \longrightarrow \widetilde{\mathbf{C}}\mathbf{omp}$ *be functors. Then the following hold:*

(1) If F *is free with basis in* $\mathbf{M}$ *and if* E *is acyclic on* $\mathbf{M}$*, then there exists an augmented natural chain map* $\tau : F \longrightarrow E$ *and any two are naturally chain homotopic.*

(2) If both E *and* F *are free with basis in* $\mathbf{M}$ *and acyclic on* $\mathbf{M}$*, then every augmented natural chain map* $\tau : F \longrightarrow E$ *is a natural chain equivalence.*

The proof of the theorem above follows from [d] Chapter VI, Propositions 11.2 and 11.7 which is the acyclic models theorem in **Comp**. Indeed, we get (1) by observing that the identity $F_{-1}C \longrightarrow E_{-1}C$ is the same as $H_0(FC) \longrightarrow H_0(EC)$ for every $C \in \mathrm{Obj}\mathbf{C}$. On the other hand, (2) follows from (1) by considering also a natural chain map $\rho : E \longrightarrow F$ and the compositions $\tau \circ \rho : E \longrightarrow E$ and $\rho \circ \tau : F \longrightarrow F$.

3 O-minimal homology

In this section we will define what is an o-minimal homology theory. This definition is from [Wo].

Below, we will denote by DTOP (resp., DCTOP) the category of definable sets (resp. definably compact sets) with continuous definable maps. By DTOPP we denote the category of pairs from DTOP that is, the objects of DTOPP are pairs (X, A) with $A \subseteq X$ in DTOP and $f : (X, A) \longrightarrow (Y, B)$ is a morphism in DTOPP if $f : X \longrightarrow Y$ is a morphism in DTOP such that $f(A) \subseteq B$. In the same way we define the category of pairs DCTOPP from DCTOP.

Below R is a ring with unit, C is either DTOPP or DCTOPP and the functor $G : C \longrightarrow C$ is the functor that sends $(X, A) \in \mathrm{Obj}C$ into $(A, \emptyset) \in \mathrm{Obj}C$ and sends $f : (X, A) \longrightarrow (Y, B) \in \mathrm{Mor}C$ into $f_{|A} : (A, \emptyset) \longrightarrow (B, \emptyset) \in \mathrm{Mor}C$.

Definition 3.1 An *o-minimal homology* (H_*, d_*) *on* C is a sequence $(H_n)_{n\in\mathbb{Z}}$ of covariant functors from C into the category of R-modules together with a sequence $(d_n)_{n\in\mathbb{Z}}$ of natural transformations $d_n : H_n \longrightarrow H_{n-1} \circ G$ such that the following axioms hold.

Homotopy Axiom. If $f, g : (X, A) \longrightarrow (Y, B) \in \mathrm{Mor}C$ and there is a definable homotopy in C between f and g, then

$$H_n(f) = H_n(g) : H_n(X, A) \longrightarrow H_n(Y, B)$$

for all $n \in \mathbb{Z}$.

Exactness Axiom. For $(X, A) \in \mathrm{Obj}C$, if $i : (A, \emptyset) \longrightarrow (X, \emptyset)$ and $j : (X, \emptyset) \longrightarrow (X, A)$ are the inclusions, then there exists a natural exact sequence

$$\cdots \longrightarrow H_n(A, \emptyset) \stackrel{H_n(i)}{\rightarrow} H_n(X, \emptyset) \stackrel{H_n(j)}{\rightarrow} H_n(X, A) \stackrel{d_n}{\rightarrow} H_{n-1}(A, \emptyset) \longrightarrow \cdots$$

(where the d_n's are induced by the natural transformations above).

Excision Axiom. For every $(X, A) \in \mathrm{Obj}C$ and every definable open subset U of X such that $\overline{U} \subseteq \mathring{A}$ and $(X - U, A - U) \in \mathrm{Obj}C$, the inclusion $(X - U, A - U) \longrightarrow (X, A)$ induces isomorphisms

$$H_n(X - U, A - U) \longrightarrow H_n(X, A)$$

for all $n \in \mathbb{Z}$.

Dimension Axiom. If X is a one point set, then $H_n(X, \emptyset) = 0$ for all $n \neq 0$. The R-module $H_0(X, \emptyset)$ is called the coefficient R-module.

These axioms are the analogues of the classical Eilenberg-Steenrod axioms for homology functors (see [dp] 4.1 and 4.2). We therefore call them the o-minimal Eilenberg-Steenrod axioms.

We will write $X \in \mathrm{Obj}C$ for $(X, \emptyset) \in \mathrm{Obj}C$, $f : X \longrightarrow Y \in \mathrm{Mor}C$ for $f : (X, \emptyset) \longrightarrow (Y, \emptyset) \in \mathrm{Mor}C$ and $H_n(X)$ for $H_n(X, \emptyset)$. Moreover, if Q is the coefficient R-module of (H_*, d_*), we will write $H_n(X, A; Q)$ instead of just $H_n(X, A)$ and the notation $H_n(X, A)$ will be used for $H_n(X, A; \mathbb{Z})$.

Two o-minimal homologies (H_*, d_*) and (H'_*, d'_*) on C with coefficient R-module Q are *isomorphic* if there is a sequence $(\tau_n)_{n\in\mathbb{Z}}$ of natural equivalences $\tau_n : H_n \longrightarrow H'_n$ such that

$$\begin{array}{ccc} H_{n+1}(X, A; Q) & \xrightarrow{d_{n+1}} & H_n(A; Q) \\ \downarrow^{\tau_{n+1}} & & \downarrow^{\tau_n} \\ H'_{n+1}(X, A; Q) & \xrightarrow{d'_{n+1}} & H'_n(A; Q) \end{array}$$

commutes for all $(X, A) \in \mathrm{Obj}C$ and for all $n \in \mathbb{Z}$.

4 O-minimal simplicial homology

Here we describe how the construction of the o-minimal simplicial homology with coefficents in $\mathbb{Z}$ is done in [Wo].

We start with the definable quasi-stratified triangulation theorem. Note that as in [vdd] Chapter VIII, the simplicial complexes considered below are not necessarily closed as in the standard case.

Suppose that $S_1, \ldots, S_k \subseteq S \subseteq N^m$ are definable sets. A *definable triangulation in N^m of S compatible with $S_1, \ldots, S_k$* is a pair (Φ, K) consisting of a simplicial complex K in N^m and a definable homeomorphism $\Phi : S \longrightarrow |K|$ such that each S_i is a union of elements of $\Phi^{-1}(K)$. Here, $|K|$ is the realization of the simplicial complex K. We say that (Φ, K) is a *stratified definable triangulation of S compatible with $S_1, \ldots, S_k$* if: $m = 0$ or $m > 0$ and there is a stratified definable triangulation (Ψ, L) of $\pi(S)$ compatible with $\pi(S_1), \ldots, \pi(S_k)$ (where $\pi : N^m \longrightarrow N^{m-1}$ is the projection onto the first $m-1$ coordinates) such that $\pi_{|\mathrm{Vert}(K)} : K \longrightarrow L$ is a simplicial map and

the diagram

$$\begin{array}{ccc} S & \stackrel{\Phi}{\rightarrow} & |K| \\ \downarrow^{\pi} & & \downarrow^{\pi} \\ \pi(S) & \stackrel{\Psi}{\rightarrow} & |L| \end{array}$$

commutes. We say that (Φ, K) is a *quasi-stratified definable triangulation of S compatible with $S_1, \ldots, S_k$* if there is a linear bijection $\alpha : N^m \longrightarrow N^m$ such that $(\alpha\Phi\alpha^{-1}, \alpha K)$ is a stratified definable triangulation of $\alpha(S)$ compatible with $\alpha(S_1), \ldots, \alpha(S_k)$.

Theorem 4.1 (Definable quasi-stratified triangulation) *Suppose that $S_1, \ldots, S_k \subseteq S \subseteq N^m$ are definable sets. Then there is a definable triangulation of S compatible with $S_1, \ldots, S_k$. Moreover, if S is bounded, then there is a quasi-stratified definable triangulation of S compatible with $S_1, \ldots, S_k$.*

The first part of this theorem is the definable triangulation theorem and appears in [vdd] Chapter 8 (2.9) and its semi-algebraic analogue in [dk]. A careful analysis of the proof of [vdd] Chapter 8 (2.9) shows that if S is bounded definable set, then there is a quasi-stratified definable triangulation of S compatible with $S_1, \ldots, S_k$.

We are now ready to describe the (relative) o-minimal simplicial chain complex which is defined, in [Wo], in exactly the same way as its classical analogue.

Let K be a closed simplicial complex in N^n. The *o-minimal simplicial chain complex* $(C_*(K), \partial_*)$ is defined in the following way. For each $l \in \mathbb{N}$, we set $C_l(K) = A_l(K)/A'_l(K)$ where $A_l(K)$ is the free abelian group generated by the set of $(l+1)$-tuples $(v_0, \ldots, v_l)$ of vertices of K such that $v_0, \ldots, v_l$ span an l-simplex in K, and $A'_l(K)$ is the subgroup generated by the elements of the form $(v_0, \ldots, v_l) - \mathrm{sgn}(\alpha)(v_{\alpha(0)}, \ldots, v_{\alpha(l)})$ where α is a permutation of $\{0, \ldots, l\}$ and $v_0, \ldots, v_l$ spans an l-simplex in K. For $l \in \mathbb{N}$, the homomorphism $\partial_l : C_l(K) \longrightarrow C_{l-1}(K)$ is the homomorphism induced by the homomorphism

$$(v_0, \ldots, v_l) \mapsto \sum_{i=0}^{l} (-1)^i (v_0, \ldots, \widehat{v}_i, \ldots, v_l) : A_l(K) \longrightarrow A_{l-1}(K)$$

where the hat on top of v_i means that the vertice v_i is omitted.

If we define $\widetilde{\partial}_0 : C_0(K) \longrightarrow \mathbb{Z}$ by $\widetilde{\partial}_0(\sum \alpha_i v_i) = \sum \alpha_i$, we obtain an augmentation $(\widetilde{C}_*(K), \widetilde{\partial}_*)$ of $(C_*(K), \partial_*)$.

Note that each $\widetilde{C}_l(K)$ is a free abelian group, and given any total order on the vertices of K, the set of all classes $\langle v_0, \ldots, v_l \rangle \in \widetilde{C}_l(K)$ such that $v_0 < \cdots < v_l$ provides a basis for $\widetilde{C}_l(K)$.

If (K, K') is a *closed simplicial pair* i.e., K and K' are closed simplicial complexes with K' a subcomplex of K, then $\widetilde{C}_*(K')$ is a subcomplex of $\widetilde{C}_*(K)$ and we define the *relative o-minimal simplicial chain complex* $(C_*(K, K'), \partial_*)$ by setting $C_*(K, K') = \widetilde{C}_*(K)/\widetilde{C}_*(K')$. Note that $C_*(K) \simeq C_*(K, \emptyset)$. We define

$$H_*(K, K') = H_*((C_*(K, K'), \partial_*)) = \mathrm{Ker}\partial_*/\mathrm{Im}\partial_{*+1}.$$

Below we will denote by CSCP the category whose objects are closed simplicial pairs in N^m for some m, and whose morphisms are the arrows $f : (K, K') \longrightarrow (L, L')$ such that $f : (|K|, |K'|) \longrightarrow (|L|, |L'|)$ is a morphism in DTOPP. By CSC we will denote the full subcategory of CSCP such that $K \in$ ObjCSC if and only if $(K, \emptyset) \in$ ObjCSCP and $f : K \longrightarrow L \in$ MorCSC if and only if $f : (K, \emptyset) \longrightarrow (L, \emptyset) \in$ MorCSCP.

Having already constructed the o-minimal simplicial chain complex associated to a closed simplicial pair we need to construct the chain map associated to a morphism of closed simplicial pairs, i.e. we need to prove the following result:

Theorem 4.2 *There is a well defined functor from the category* CSCP *into* **hComp** *sending* $(K, K') \in$ ObjCSCP *into* $(C_*(K, K'), \partial_*)$ *and sending* $f : (K, K') \longrightarrow (L, L') \in$ MorCSCP *into* $f_\sharp : C_*(K, K') \longrightarrow C_*(L, L')$.

In the classical case, it is at this point that simplicial approximation theorem plays a role (see [dp] 5.4). As we pointed out already, this approach fails in the o-minimal context because the field $(N, 0, 1, <, +, \cdot)$ may be non-archemedian. The second author handled this difficulty in [Wo] in the way that we will now present. Roughly, we start by proving Theorem 4.2 for compatible definable maps $f : K \longrightarrow L \in$ MorCSC (see Definition 4.3 and Lemma 4.5 below) and then use definable stratified triangulations to generalize the construction to arbitrary definable maps $f : K \longrightarrow L \in$ MorCSC.

Definition 4.3 For a closed simplicial complex K, let $\mathbf{A}(K)$ denote the category whose objects are closed simplicial subcomplexes of K and whose morphisms are the inclusion maps.

Let $\widetilde{C}_*^L : \mathbf{A}(L) \longrightarrow \widetilde{\mathbf{C}}\mathbf{omp}$ be the functor which sends a closed subcomplex L' of L to the augmented chain complex $\widetilde{C}_*(L')$ and sends an inclusion $L'' \longrightarrow L'$ to the inclusion $\widetilde{C}_*(L'') \longrightarrow \widetilde{C}_*(L')$.

We say that $f : K \longrightarrow L \in \mathrm{MorCSC}$ is *compatible* if, for each simplex $\tau \in K$, there is a simplex $\sigma \in L$ such that $f(|\tau|) \subseteq |\sigma|$. For each compatible $f : K \longrightarrow L \in \mathrm{MorCSC}$, we define the functor $\mathbf{A}_f : \mathbf{A}(L) \longrightarrow \mathbf{A}(K)$ by $\mathbf{A}_f(L') = \{\tau \in K : f(|\tau|) \subseteq |L'|\}$.

Lemma 4.4 *If $f : K \longrightarrow L \in \mathrm{MorCSC}$ is a compatible definable map, then there is an augmented natural chain map $\widetilde{C}_*^K \circ \mathbf{A}_f \longrightarrow \widetilde{C}_*^L$ and any two are naturally chain homotopic.*

Proof. The set $\mathbf{M}_L = \{\overline{\sigma} : \sigma \in L\}$ is a set of models for $\mathbf{A}(L)$ and clearly $\widetilde{C}_n^L$ is acyclic on $\mathbf{M}_L$.

Fix a total ordering of the vertices of K. For $n \in \mathbb{N}$, denote by I_n^K the basis of $\widetilde{C}_n(K)$ consisting of all classes $\langle v_0, \ldots, v_n \rangle$ such that $v_0 < \cdots < v_n$ and identify $\alpha \in I_n^K$ with the simplex determined by α. For each $\alpha \in I_n^K$, let τ_α is the unique simplex of L such that $f(|\alpha|) \subseteq |\tau_\alpha|$. Then $(\overline{\tau}_\alpha)_{\alpha \in I_n^K}$ is a family of models in $\mathbf{M}_L$ and $\{\, \widetilde{C}_n^K \circ \mathbf{A}_f(\ell)(\overline{\tau}_\alpha) : \alpha \in I_n^K$ and $\ell : \overline{\tau}_\alpha \longrightarrow S$ an inclusion $\}$ is a basis of $\widetilde{C}_n^K \circ \mathbf{A}_f(S)$ for every $S \in \mathrm{Obj}\mathbf{A}(L)$. So $\widetilde{C}_*^K \circ \mathbf{A}_f$ is free with basis in $\mathbf{M}_L$. Now apply the method of acyclic models (Theorem 2.4). $\square$

Here is a consequence of Lemma 4.4:

Lemma 4.5 *For every compatible definable map $f : K \longrightarrow L \in \mathrm{MorCSC}$ there exists a chain map $f_\sharp : \widetilde{C}_*(K) \longrightarrow \widetilde{C}_*(L)$ which is the homotopy class of chain maps $\widetilde{C}_*(K) \longrightarrow \widetilde{C}_*(L)$ determined by augmented natural chain maps $\widetilde{C}_*^K \circ \mathbf{A}_f \longrightarrow \widetilde{C}_*^L$ given by Lemma 4.4. Moreover, if $h : K \longrightarrow L$ and $g : L \longrightarrow M$ in MorCSC are compatible definable maps, then $(g \circ h)_\sharp = g_\sharp \circ h_\sharp$.*

Proof. Let ρ (resp., λ) be an augmented natural chain map $\widetilde{C}_*^K \circ \mathbf{A}_h \longrightarrow \widetilde{C}_*^L$ (resp., $\widetilde{C}_*^L \circ \mathbf{A}_g \longrightarrow \widetilde{C}_*^M$) given by Lemma 4.4. For each $S \in \mathrm{Obj}\mathbf{A}(M)$

define the chain map $\mu_S : \widetilde{C}_*^K \circ \mathbf{A}_{g\circ h}(S) \longrightarrow \widetilde{C}_*^L \circ \mathbf{A}_g(S)$ by setting $\mu_S = \lambda_{\mathbf{A}_g(M)}$. Then μ is an augmented natural chain map $\widetilde{C}_*^K \circ \mathbf{A}_{g\circ h} \longrightarrow \widetilde{C}_*^L \circ \mathbf{A}_g$, and the composition $\lambda \circ \mu$ is an augmented natural chain map $\widetilde{C}_*^K \circ \mathbf{A}_{g\circ h} \longrightarrow \widetilde{C}_*^M$. Since $\lambda_M \circ \rho_L = (\lambda \circ \mu)_M$ it follows that $[\lambda_M] \circ [\rho_L] = [(\lambda \circ \mu)_M]$. Thus $g_\sharp \circ h_\sharp = (g \circ h)_\sharp$. $\square$

Our next goal is to prove the following result which will allow us to define a chain map $f_\sharp : \widetilde{C}_*(K) \longrightarrow \widetilde{C}_*(L)$ for $f : K \longrightarrow L \in$ MorCSC a definable map not necessarily compatible.

Lemma 4.6 *Let K be a closed simplicial complex and let (Ψ, L) be a quasi-stratified triangulation of $|K|$ such that $\Psi^{-1} : L \longrightarrow K$ is compatible. Then every augmented natural chain map $\widetilde{C}_*^L \circ \mathbf{A}_{\Psi^{-1}} \longrightarrow \widetilde{C}_*^K$ is a natural chain equivalence.*

Lemma 4.6 is an immediate consequence of the following result:

Lemma 4.7 *Let X be a nonempty, closed, bounded, convex, definable subset of N^n and (Ψ, L) a stratified triangulation of X. Then $\widetilde{C}_*(L)$ is an acyclic chain complex.*

Proof of Lemma 4.6: Let K be a closed simplicial complex and let (Ψ, L) be a quasi-stratified triangulation of $|K|$ such that $\Psi^{-1} : L \longrightarrow K$ is compatible. As in the proof of Lemma 4.4, $\widetilde{C}_*^K$ is acyclic on $\mathbf{M}_K = \{\overline{\sigma} : \sigma \in K\}$ and $\widetilde{C}_*^L \circ \mathbf{A}_{\Psi^{-1}}$ and $\widetilde{C}_*^K = \widetilde{C}_*^K \circ \mathbf{A}_{1_K}$ are free with basis in $\mathbf{M}_K$. For each $\overline{\sigma} \in \mathbf{M}_K$, $(\Psi_{||\overline{\sigma}|}, \mathbf{A}_{\Psi^{-1}}(\overline{\sigma}))$ is a quasi-stratified triangulation of the convex set $|\overline{\sigma}|$. By Lemma 4.7, $\widetilde{C}_*^L \circ \mathbf{A}_{\Psi^{-1}}$ is acylcic on $\mathbf{M}_K$. Hence, by the method of acyclic models (Theorem 2.4) we conclude that every augmented natural chain map $\widetilde{C}_*^L \circ \mathbf{A}_{\Psi^{-1}} \longrightarrow \widetilde{C}_*^K$ is a natural chain equivalence as required. $\square$

We now go on to prove Lemma 4.7. Below, by $\pi : N^{n+1} \longrightarrow N^n$ we will denote the projection map onto the first n coordinates.

Proof of Lemma 4.7: A pair (K, L) of simplicial complexes is said to be a *projective pair* if for some $n \in \mathbb{N}$, K is a nonempty, closed simplicial complex in N^n, L is a closed simplicial complex in N^{n+1}, the restriction $\pi_{|\mathrm{Vert(L)}} : L \longrightarrow K$ is a simplicial map and $\pi(|L|) = |K|$.

We say that a projective pair (K, L) of simplicial complexes has *convex fibers* if, for each $(x_1, \ldots, x_n) \in |K|$, the set $\{x \in N : (x_1, \ldots, x_n, x) \in |L|\}$ is an interval $[a, b]$ in N, with $-\infty < a \leq b < +\infty$.

Let (K, L) be a projective pair of simplicial complexes. For each $\sigma \in K$, we let L_σ denote the set $\{\beta \in L : \pi(|\beta|) = |\sigma|\}$. Each set L_σ is linearly ordered by defining $\beta_1 \prec \beta_2$ if, for all $x = (x_1, \ldots, x_{n+1}) \in \beta_1$ and $y = (y_1, \ldots, y_{n+1}) \in \beta_2$ with $\pi(x) = \pi(y)$, we have $x_{n+1} < y_{n+1}$. A simplex $\rho \in L$ is called *thin* if $\dim\rho = \dim\pi\rho$, and a simplex $\lambda \in L$ is called *thick* if $\dim\lambda = \dim\pi\lambda + 1$.

Claim 4.8 *Let (K, L) be a projective pair of simplicial complexes with convex fibers, and let $\sigma \in K$. Then L_σ is of the form $\{\alpha_0, \beta_1, \alpha_1, \beta_2, \ldots, \beta_k, \alpha_k\}$ for some $k \geq 0$, with $\alpha_0 \prec \beta_1 \prec \alpha_1 \prec \beta_2 \prec \ldots \prec \beta_k \prec \alpha_k$, with all α_i thin and all β_i thick, and with α_{i-1} and α_i faces of β_i for $0 < i \leq k$.*

Proof. The set L_σ is bounded above and below by the thin simplices since L is closed. The rest of the claim can be deduced from the fact that: (i) if $\alpha', \alpha'' \in L_\sigma$ are thin and $\alpha' \prec \alpha''$, then there exists a thick $\beta \in L_\sigma$ with $\alpha' \prec \beta \prec \alpha''$; (ii) if $\beta', \beta'' \in L_\sigma$ are thick and $\beta' \prec \beta''$, then there exists a thin $\alpha \in L_\sigma$ with $\beta' \prec \alpha \prec \beta''$. $\square$

Claim 4.9 *Let (K, L) be a projective pair of simplicial complexes with convex fibers and suppose that $\sigma \in K$ is such that $\overline{\sigma} = K$. Then $\widetilde{C}_*(L)$ is acyclic.*

Proof. This is proved by induction on the number of simplices of L. The number of simplices of K is $2^{\dim\sigma+1} - 1$. Since by Claim 4.8 the cardinality of L_τ is odd for each $\tau \in K$, it follows that L has an odd number of simplices.

In the base case of the induction, L has the same number of simplices as K. It follows that $L = \overline{\alpha}$ for some $\alpha \in L$ with $\dim\alpha = \dim\sigma$. In this case, $\widetilde{C}_*(L)$ is clearly acyclic.

So suppose that the number of simplices in L is greater than in K. Then $L_\sigma = \{\alpha_0, \beta_1, \alpha_1, \beta_2, \ldots, \beta_k, \alpha_k\}$ as in Claim 4.8. We consider two cases.

Case (1): $k > 0$.

Let $M = L - \{\alpha_k, \beta_k\}$. First, note that (K, M) is a projective pair with convex fibers. Since M has two fewer simplices than L, it follows from the induction hypothesis that $\widetilde{C}_*(M)$ is acyclic. Second, note that β_k is the only

simplex $\beta \in L$ such that $\alpha_k \prec \beta$. By the theorem on removing cells (Theorem 2.1), $\widetilde{C}_*(M)$ is an adequate subcomplex of $\widetilde{C}_*(L)$, and hence $\widetilde{C}_*(L)$ is also acyclic.

Case (2): $k = 0$, that is $L_\sigma = \{\alpha\}$, where α is a thin simplex of L with $\dim\alpha = \dim\sigma$.

Note that: (i) $\overline{\alpha}$ is a subcomplex of L; (ii) β is thin for each $\beta \in \overline{\alpha}$; and for each $\tau \in \overline{\sigma}$, there exists a unique $\beta \in \overline{\alpha}$ such that $\beta \in L_\tau$. Let τ be a simplex in K of maximal dimension such that L_τ contains more than one simplex. By Claim 4.8, L_τ is of the form $\{\rho_0, \lambda_1, \rho_1, \lambda_2, \ldots, \lambda_l, \rho_l\}$ for some $l > 0$. Let $i \in \{0, \ldots, l\}$ be the unique index such that $\rho_i \in \overline{\alpha}$. Note that for all simplices $\beta \in L - L_\tau$, $\overline{|\beta|} \cap |L_\tau| \neq \emptyset$ implies that $\tau \prec \pi\beta$ and $\pi\beta \prec \sigma$ or $\pi\beta = \sigma$, and hence $\overline{|\beta|} \cap |L_\tau| = \rho_i$.

If $i = 0$, then λ_l is the only simplex $\beta \in L$ such that $\rho_l \prec \beta$. Let $M = L - \{\rho_l, \lambda_l\}$. Then (K, M) is a projective pair with convex fibers and, once again, since M has two fewer simplices than L, $\widetilde{C}_*(M)$ is acyclic by induction. By the theorem on removing cells (Theorem 2.1), $\widetilde{C}_*(M)$ is an adequate subcomplex of $\widetilde{C}_*(L)$, and hence $\widetilde{C}_*(L)$ is also acyclic.

The case $i > 0$ is similar. Simply let $M = L - \{\rho_0, \lambda_1\}$. □

We are now ready to finish the proof of Lemma 4.7. We use induction on n. The case $n = 0$ is trivial. So suppose that $n > 0$. Then (Ψ, L) is a lifting of a stratified triangulation (Φ, K) of $\pi(X)$. It follows that (K, L) is a projective pair and, in particular, it follows from the commutativity property of liftings and the convexity of X that (K, L) has convex fibers. Note that $\pi_L = \pi_{||L|} : L \longrightarrow K$ is compatible, so by Lemma 4.4 there is a natural chain map $\widetilde{C}_*^L \circ \mathbf{A}_{\pi_L} \longrightarrow \widetilde{C}_*^K$. By induction, $\widetilde{C}_*(K)$ is acyclic. Thus it suffices to show that the natural chain map $\widetilde{C}_*^L \circ \mathbf{A}_{\pi_L} \longrightarrow \widetilde{C}_*^K$ is a natural chain equivalence.

As in the proof of Lemma 4.4, $\widetilde{C}_*^K$ is acyclic on $\mathbf{M}_K = \{\overline{\sigma} : \sigma \in K\}$ and $\widetilde{C}_*^L \circ \mathbf{A}_{\pi_L}$ and $\widetilde{C}_*^K = \widetilde{C}_*^K \circ \mathbf{A}_{1_K}$ are free with basis in $\mathbf{M}_K$. By Claim 4.9, $\widetilde{C}_*(\pi_L^{-1}(|\overline{\sigma}|))$ is an acyclic chain complex for each $\sigma \in K$, i.e. $\widetilde{C}_*^L \circ \mathbf{A}_{\pi_L}$ is acyclic on $\mathbf{M}_K$. The method of acyclic models (Theorem 2.4) completes the proof. □

Finally we are ready to prove Theorem 4.2.

Proof of Theorem 4.2: Suppose $f : K \longrightarrow L \in$ MorCSC is a definable map not necessarily compatible. Let (Φ, M) be a quasi-stratified triangulation of $|K|$ such that $\Phi^{-1} : M \longrightarrow K$ and $f \circ \Phi^{-1} : M \longrightarrow L$ are compatible. Define $f_\sharp^{(\Phi,M)}$ to be $(f \circ \Phi^{-1})_\sharp \circ (\Phi^{-1})_\sharp^{-1} : \widetilde{C}_*(K) \longrightarrow \widetilde{C}_*(L)$.

Claim 4.10 *Suppose that $f : K \longrightarrow L \in$ MorCSC and (Φ, M) is as above. Let (Ψ, N) be a quasi-stratified triangulation of $|K|$ such that $\Phi \circ \Psi^{-1} : N \longrightarrow M$ is compatible. Then $f_\sharp^{(\Phi,M)} = f_\sharp^{(\Psi,N)}$.*

Proof.

$$\begin{aligned}
&f_\sharp^{(\Psi,N)} = (f \circ \Psi^{-1})_\sharp \circ ((\Psi^{-1})_\sharp)^{-1} \\
&= (f \circ \Phi^{-1} \circ \Phi \circ \Psi^{-1})_\sharp \circ ((\Phi^{-1} \circ \Phi \circ \Psi^{-1})_\sharp)^{-1} \\
&= (f \circ \Phi^{-1})_\sharp \circ (\Phi \circ \Psi^{-1})_\sharp \circ ((\Phi \circ \Psi^{-1})_\sharp)^{-1} \circ ((\Phi^{-1})_\sharp)^{-1} \\
&= f_\sharp^{(\Phi,M)}.
\end{aligned}$$

□

It follows easily from Claim 4.10 that $f_\sharp^{(\Phi,M)}$ is independent of the triangulation. In light of this, we will use the simpler notation $f_\sharp$ to denote the homotopy class of chain maps determined by f.

Claim 4.11 *Let $f : K \longrightarrow K'$ and $g : K' \longrightarrow K''$ be morphisms of* CSC. *Then $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$.*

Proof. Let (Φ, M) be an appropriate triangulation of $|K'|$ such that $g_\sharp = g_\sharp^{(\Phi,M)}$. By the definable triangulation theorem, there is a quasi-stratified triangulation (Ψ, N) of $|K|$ such that $\Phi \circ f \circ \Psi^{-1} : N \longrightarrow M$ is compatible. We use the fact that $\sharp$ distributes over composition of compatible maps.

$$\begin{aligned}
&(g \circ f)_\sharp = (g \circ f)_\sharp^{(\Psi,N)} \\
&= (g \circ f \circ \Psi^{-1})_\sharp \circ ((\Psi^{-1})_\sharp)^{-1} \\
&= (g \circ \Phi^{-1} \circ \Phi \circ f \circ \Psi^{-1})_\sharp \circ ((\Psi^{-1})_\sharp)^{-1} \\
&= (g \circ \Phi^{-1})_\sharp \circ ((\Phi^{-1})_\sharp)^{-1} \circ (\Phi^{-1})_\sharp \circ (\Phi \circ f \circ \Psi^{-1})_\sharp \circ ((\Psi^{-1})_\sharp)^{-1} \\
&= g_\sharp^{(\Phi,M)} \circ f_\sharp^{(\Psi,N)} = g_\sharp \circ f_\sharp.
\end{aligned}$$

□

For any $f : (K, K') \longrightarrow (L, L')$, there is a representative τ_K of the homotopy class $f_\sharp$ such that $\tau_K(\widetilde{C}_*(K')) \subseteq \widetilde{C}_*(L')$. Therefore $f_\sharp$ induces a homotopy class of chain maps $C_*(K, K') \longrightarrow C_*(L, L')$. Thus we have established one of the main results of this section, namely Theorem 4.2. $\square$

With Theorem 4.2 available, the o-minimal simplicial homology is defined in [Wo] in the following way.

Definition 4.12 For $(X, A) \in$ ObjDCTOPP, let $T(X, A)$ denote the set of all definable triangulations (Φ, K) of X which respects A, and for each $(\Phi, K) \in T(X, A)$, let K' denote the closed subcomplex of K such that $\Phi(A) = |K'|$. The *n-th simplicial homology group* $H_n(X, A)$ is defined to be the subgroup of $\prod_{(\Phi,K)\in T(X,A)} H_n(K, K')$ consisting of all elements α such that, for all $(\Phi_1, K_1), (\Phi_2, K_2) \in T(X, A)$, we have $\alpha_{(\Phi_2,K_2)} = H_n(\Phi_2 \circ \Phi_1^{-1})(\alpha_{(\Phi_1,K_1)})$.

Given $f : (X, A) \longrightarrow (Y, B) \in$ MorDCTOPP we define n-th induced homomorphism $H_n(f) : H_n(X, A) \longrightarrow H_n(Y, B)$ such that if $H_n(f)(\alpha) = \beta$, then for all $(\Phi, K) \in T(X, A)$ and $(\Psi, L) \in T(Y, B)$, we have $\beta_{(\Psi,L)} = H_n((\Psi \circ f \circ \Phi^{-1})_\sharp)(\alpha_{(\Phi,K)})$.

The verification of the Eilenberg-Steenrod axioms is now easy. In fact, the arguments are similar to their analogues for the classical topological simplicial homology.

Theorem 4.13 *There is an o-minimal homology (H_*, d_*) on* DCTOPP *with coefficients in $\mathbb{Z}$ such that if $(X, A) \in$* ObjDCTOPP*, (Φ, K) is a definable triangulation of X which respects A and K' is the subcomplex of K such that $|K'| = \Phi(A)$, then we have isomorphisms $\pi_n^{(\Phi,K)} : H_n(X, A) \longrightarrow H_n(K, K')$ for all $n \in \mathbb{Z}$. Also, if $f : (X, A) \longrightarrow (Y, B) \in$* MorDCTOPP *and (Ψ, L) is a definable triangulation of Y respecting B and such that $|L'| = \Psi(B)$, then following diagram*

$$\begin{array}{ccc} H_n(X, A) & \xrightarrow{H_n(f)} & H_n(Y, B) \\ \downarrow{\scriptstyle \pi_n^{(\Phi,K)}} & & \downarrow{\scriptstyle \pi_n^{(\Psi,L)}} \\ H_n(K, K') & \xrightarrow{H_n((\Psi\circ f\circ\Phi^{-1})_\sharp)} & H_n(L, L') \end{array}$$

is commutative.

Proof. By the above, it follows easily that each H_n defines a functor from DCTOPP into the category of abelian groups. We now verify the o-minimal Eilenberg-Steenrod axioms for the sequence of functors H_* we have just defined.

Proof of the homotopy axiom. If $F : (X \times [a,b], A \times [a,b]) \longrightarrow (Y, B)$ is a definable homotopy in DCTOPP between $f_a, f_b : (X, A) \longrightarrow (Y, B) \in$ MorDCTOPP, then $f_c = F \circ g_c$ where for $c = a, b$, $g_c : (X, A) \longrightarrow (X \times [a,b], A \times [a,b]) \in$ MorDCTOPP is given by $g_c(x) = (x, c)$. So to prove the homotopy axiom, it is clearly enough to show that $H_*(g_a) = H_*(g_b)$. Let $(K, K') \in T(X, A)$. Then, we have a canonical $(M, M') \in T(X \times [a,b], A \times [a,b])$ associated to (K, K') such that the induced morphisms $g_c : K \longrightarrow M$ $(c = a, b)$ and $\pi : M \longrightarrow K$ (induced by the projection $\pi : X \times [a,b] \longrightarrow X$) in MorCSC are compatible. Since the morphisms $g_c : K \longrightarrow M$ are compatible, by Lemma 4.4, they induce augmented natural chain maps $\tau^c : \widetilde{C}_*^K \circ \mathbf{A}_{g_c} \longrightarrow \widetilde{C}_*^M$, which induce chain maps $(g_c)_\sharp : C_*(K, K') \longrightarrow C_*(M, M')$. Since, by definition $H_*(g_c) = H_*((g_c)_\sharp)$ it is enough to show that $(g_a)_\sharp$ and $(g_b)_\sharp$ are chain homotopic.

Let $\mathbf{B}$ be the subcategory of $\mathbf{A}(M)$ consisting of subcomplexes of M of the form $\mathbf{A}_\pi(S)$, where S is a closed subcomplex of K. It is easy to see that: (1) the set $\mathbf{M} = \{\mathbf{A}_\pi(\overline{\sigma}) : \sigma \in K\}$ is a set of models for $\mathbf{B}$; (2) each τ^c restricts to an augmented natural chain map $\tau^c_{|\mathbf{B}} : \widetilde{C}_*^K \circ \mathbf{A}_{g_c|\mathbf{B}} \longrightarrow (\widetilde{C}_*^M)_{|\mathbf{B}}$ and (3) $\widetilde{C}_*^K \circ \mathbf{A}_{g_c|\mathbf{B}}$ is free with basis in $\mathbf{M}$ and $(\widetilde{C}_*^M)_{|\mathbf{B}}$ is acyclic on $\mathbf{M}$ (by Claim 4.9, since $(\overline{\sigma}, \mathbf{A}_\pi(\overline{\sigma}))$ is a projective pair of simplicial complexes with convex fibers). Therefore, by the method of acyclic models (Theorem 2.4) , there is a natural chain homotopy $\tau^a_{|\mathbf{B}} \simeq \tau^b_{|\mathbf{B}}$ which induces the desired chain homotopy $(g_a)_\sharp \simeq (g_b)_\sharp$.

Proof of the exactness axiom. Suppose that $(X, A) \in$ ObjDCTOPP and $(K, K') \in T(X, A)$. Then we have a short exact sequence of chain maps

$$0 \to C_*(K') \to C_*(K) \to C_*(K, K') \to 0$$

which, by basic homological algebra ([sp] Chapter 4, Section 5, Lemma 3 and Theorem 4), gives connecting homomorphisms $d_n^{K,K'} : H_n(K, K') \longrightarrow H_{n-1}(K')$ making the induced sequence

$$\cdots \longrightarrow H_n(K', \emptyset) \to H_n(K, \emptyset) \to H_n(K, K') \xrightarrow{d_n^{K,K'}} H_{n-1}(K', \emptyset) \longrightarrow \cdots$$

exact. Since these connecting homomorphisms are induced by natural transformations in the category of short exact sequences of chain maps ([sp] Chapter 4, Section 5, Lemma 3 and Theorem 4), they determine natural transformations $d_n : H_n \longrightarrow H_{n-1} \circ G$ satisfying the exactness axiom by requiring that the following diagram

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{d_n^{X,A}} & H_{n-1}(A) \\ \downarrow^{\pi^{(K,K')}} & & \downarrow^{\pi^{(K',\emptyset)}} \\ H_n(K,K') & \xrightarrow{d_n^{K,K'}} & H_{n-1}(K') \end{array}$$

commutes.

Proof of the excision axiom. Let $(X,A) \in$ ObjDCTOPP and let $U \in$ ObjDTOP be an open subset of X such that $\overline{U} \subseteq \mathring{A}$. Let (K,Φ) be a definable triangulation of X compatible with A, $X-A$, U and $X-U$, and let K' and K'' be the simplicial subcomplexes of K such that $A = |K'|$ and $U = |K''|$. Let $L = K - K''$ and $L' = K' - K''$. Then $(L, \Phi_{||L|})$ and $(L', \Phi_{|L'|})$ are definable triangulations such that $X - U = |L|$ and $A - U = |L'|$.

Let G_n be the free abelian group generated by those basis elements $\langle v_0, \ldots, v_n \rangle$ of $\widetilde{C}_n(K)$ such that $(v_0, \ldots, v_n)$ is a simplex of K''. Then G_n is a subgroup of $\widetilde{C}_n(K)$ and of $\widetilde{C}_n(K')$ (because $\overline{U} \subseteq \mathring{A}$) such that $\widetilde{C}_n(K) = \widetilde{C}_n(L) \oplus G_n$ and $\widetilde{C}_n(K') = \widetilde{C}_n(L') \oplus G_n$. Thus $\frac{\widetilde{C}_n(K)}{\widetilde{C}_n(K')} = \frac{\widetilde{C}_n(L) \oplus G_n}{\widetilde{C}_n(L') \oplus G_n} \simeq \frac{\widetilde{C}_n(L)}{\widetilde{C}_n(L')}$, the inclusion $(L, L') \longrightarrow (K, K')$ induces a chain equivalence $C_*(L, L') \longrightarrow C_*(K, K')$ and hence isomorphisms in homology.

Proof of the dimension axiom. Simply note that if K is a simplicial complex consisting of a single 0-simplex, then $H_n(K) = 0$ for all $n \neq 0$ and $H_0(K) \simeq \mathbb{Z}$. $\square$

5 O-minimal singular homology

Here we include the proof, from [Wo], of the existence of the o-minimal singular homology with coefficients in $\mathbb{Z}$. We start with the o-minimal singular chain complex which is defined just like in the topological setting but replacing the topological category by the definable category.

The *standard n-simplex* Δ^n in DCTOP is the convex hull of the standard basis vectors $e_0, \ldots, e_n$ in N^{n+1}. Let the standard (-1)-simplex Δ^{-1} in DCTOP be the empty set.

Let $X \in$ ObjDTOP. For $n \geq -1$, we define $\widetilde{S}_n(X)$ to be the free abelian group on the set of definable continuous maps $\sigma : \Delta^n \longrightarrow X$. For $n < -1$, we set $\widetilde{S}_n(X) = 0$. Note that $\widetilde{S}_{-1}(X) = \mathbb{Z}$. The elements of $\widetilde{S}_n(X)$ are called the definable n-chains.

For $n > 0$ and $0 \leq i \leq n$ let $\epsilon_i^n : \Delta^{n-1} \longrightarrow \Delta^n$ be the continuous definable map given by $\epsilon_i^n(\sum_{j=0}^{n-1} a_j e_j) = \sum_{j<i} a_j e_j + \sum_{j \geq i}^{n-1} a_j e_{j+1}$. Let $\epsilon_0^0 : \Delta^{-1} \longrightarrow \Delta^0$ be the unique map. We define the boundary homomorphism $\partial_n : \widetilde{S}_n(X) \longrightarrow \widetilde{S}_{n-1}(X)$ to be the trivial homomorphism for $n < 0$ and for $n \geq 0$, ∂_n is given on basis elements by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ \epsilon_i^n.$$

One verifies that $\partial^2 = 0$ and so $(\widetilde{S}_*(X), \partial_*)$ is a chain complex, the *augmented o-minimal singular chain complex.*

For $f : X \longrightarrow Y \in$ MorDTOP, we have an induced chain map $f_\sharp : \widetilde{S}_*(X) \longrightarrow \widetilde{S}_*(Y)$ given on the basis elements of $\widetilde{S}_*(X)$ by $f_\sharp(\sigma) = f \circ \sigma$.

Given an element $\sigma = \sum_{i=1}^{k} n_i \sigma_i$ of $S_m(X)$ where $\sigma_i : \Delta^m \longrightarrow X$ are distinct definable continuous maps and the n_i's are non zero, the *support* Imσ of σ is by definiton the definable subset $\cup\{\sigma_i(\Delta^m) : i = 1, \ldots, k\}$ of X.

If $(X, A) \in$ ObjDTOPP, then the *relative o-minimal singular chain complex* $(S_*(X, A), \partial_*)$ is the quotient chain complex $(\widetilde{S}_*(X)/\widetilde{S}_*(A), \partial_*)$, and the o-minimal singular chain complex $(S_*(X), \partial_*)$ is $(S_*(X, \emptyset), \partial_*)$.

For $f : (X, A) \longrightarrow (Y, B) \in$ MorDTOPP, we have an induced chain map $f_\sharp : S_*(X, A) \longrightarrow S_*(Y, B)$. We define

$$H_n(X, A) = H_n(S_*(X, A), \partial_*) = \mathrm{Ker}\partial_*/\mathrm{Im}\partial_{*+1},$$

$H_n(f) = H_n(f_\sharp)$ and set $H_n(X) = H_n(X, \emptyset)$.

Having defined the o-minimal singular homology groups, it remains to show that they verify the o-minimal Eilenberg-Steenrod axioms. This will be done below and was originally done by the second author in [Wo]. The argument for the o-minimal singular exactness and dimension axiom is similar to the classical case, so we refer the reader to [d] Chapter III (3.2) and (4.2).

For the o-minimal singular homotopy axiom we require o-minimal versions of classical contructions. These will play an important role later in the proof of the o-minimal singular excision axiom.

Let $X \subseteq N^m$ be a convex definable set and let $p \in X$. The *cone construction over p in X* is a sequence of homomorphisms $z \mapsto p.z$: $\widetilde{S}_*(X) \longrightarrow \widetilde{S}_{*+1}(X)$ defined as follows: For $n < -1$, $p.$ is defined as the trivial homomorphism and for $n \geq -1$ and a basis element σ, we set

$$p.\sigma(\sum_{i=0}^{n+1} t_i e_i) = \begin{cases} p & \text{if } t_0 = 1, \\ t_0 p + (1-t_0)\sigma(\sum_{i=1}^{n+1} \frac{t_i}{1-t_0} e_i) & \text{if } t_0 \neq 1. \end{cases}$$

Given objects X, Y and Z of DTOP, let $F(X,Y)$ denote the free abelian group on the set of all definable continuous maps from X into Y. Given $\alpha = \sum_i s_i \alpha_i \in F(X,Y)$ and $\beta = \sum_j t_j \beta_j \in F(Y,Z)$, we define the sharp operator by

$$\beta_\sharp \alpha = \sum_{i,j} s_i t_j (\beta_j \circ \alpha_i) \in F(X,Z).$$

Note that $\sharp$ is associative, and since $\widetilde{S}_n(X) = F(\Delta^n, X)$, every $z \in \widetilde{S}_*(X)$ yields a chain map $z_\sharp : \widetilde{S}_*(\Delta^n) \longrightarrow \widetilde{S}_*(X)$.

Lemma 5.1 *Suppose that $X \subseteq N^m$ is a convex definable set. Let $p \in X$ and $z \in \widetilde{S}_n(X)$. Then $(p.z)_\sharp \epsilon_0^{n+1} = z$ and $(p.z)_\sharp \epsilon_i^{n+1} = p.(z_\sharp \epsilon_{i-1}^n)$ for $0 < i \leq n$. Furthermore, $p.(\partial z) + \partial(p.z) = z$.*

The proof of Lemma 5.1 is a simple computation using the definitions. For details see its classical analogue in [d] Chapter III, (4.8) and (4.9).

The cone construction is used as in the topological case to prove the following lemma. See [d] Chapter III, Proposition 4.6.

Lemma 5.2 *If $X \subseteq N^m$ be a convex definable set, then $\widetilde{S}_*(X)$ is an acyclic chain complex.*

Proof. It follows from Lemma 5.1 that $1_{\widetilde{S}_n(X)} - 0_{\widetilde{S}_n(X)} = \partial_{n+1} \circ p_n + p_{n+1} \circ \partial_n$. Thus p is a chain homotopy between the identity $1_* : \widetilde{S}_*(X) \longrightarrow \widetilde{S}_*(X)$ and the zero map $0_* : \widetilde{S}_*(X) \longrightarrow \widetilde{S}_*(X)$. Thus $H_n(1_*) = H_n(0_*)$ and $\widetilde{H}_n(X) = 0$ for all $n \in \mathbb{Z}$. $\square$

Lemma 5.3 *Consider the set of models* $\mathbf{M} = \{\Delta^n : n \in \mathbb{Z}\}$ *for* DTOP. *Then the functor* $\widetilde{S}_* : \mathrm{DTOP} \longrightarrow \widetilde{\mathbf{C}}\mathbf{omp}$ *is acyclic on* $\mathbf{M}$ *and free with basis in* $\mathbf{M}$.

Proof. By Lemma 5.2, $\widetilde{S}_*$ is acyclic on $\mathbf{M}$. Let $n \in \mathbb{Z}$. We shall construct a basis x^n of $\widetilde{S}_n$ in $\mathbf{M}$. Let x^n be the identity map $1_{\Delta^n} : \Delta^n \longrightarrow \Delta^n \in \widetilde{S}_n(\Delta^n)$. Let $X \in \mathrm{ObjDTOP}$. Note that $\{\widetilde{S}_n(\sigma)(x^n) : \sigma : \Delta^n \longrightarrow X \in \mathrm{MorDTOP}\} = \{\sigma : \Delta^n \longrightarrow X \in \mathrm{MorDTOP}\}$, which is a basis of $\widetilde{S}_n(X)$ by definition. $\square$

By Lemma 5.2, the functor $\mathrm{DTOP} \longrightarrow \widetilde{\mathbf{C}}\mathbf{omp} : X \mapsto \widetilde{S}_*([a,b] \times X)$ is acyclic on $\mathbf{M}$, so by Lemma 5.3 and the method of acyclic models (Theorem 2.4), there is a natural chain map from the functor $\widetilde{S}_*$ into this functor which is unique up to natural chain homotopy. From this and arguing as in the proof of 5.1 in [d] page 39 we verify the o-minimal singular homotopy axiom.

We now go on to prove the o-minimal singular excision axiom. Since every definable set equipped with its topology is usually totally disconnected and never locally compact, the standard proof of the topological singular excision axiom based on repeated barycentric subdivisions and the Lebesgue number property fails. Compare with [d] Chapter III Proposition 6.3 and 7.3.

Definition 5.4 Let $X \in \mathrm{ObjDTOP}$. The *barycentric subdivision* $\mathrm{Sd}_n : \widetilde{S}_n(X) \longrightarrow \widetilde{S}_n(X)$ is defined as follows: for $n < -1$, $\mathrm{Sd}_n : \widetilde{S}_n(X) \longrightarrow \widetilde{S}_n(X)$ is the trivial homomorphism, $\mathrm{Sd}_{-1} : \widetilde{S}_{-1}(X) \longrightarrow \widetilde{S}_{-1}(X)$ is the identity and for $n \geq 0$, we set $\mathrm{Sd}_n(z) = z_\sharp(b_n.\mathrm{Sd}_{n-1}\partial 1_{\Delta^n})$, where $b_n = \frac{1}{n+1}(e_0 + \cdots + e_n)$ is the barycentre of Δ^n.

Note that Sd_0 is the identity and Sd_n is natural i.e., it commutes with $f_\sharp : \widetilde{S}_n(X) \longrightarrow \widetilde{S}_n(Y)$.

Lemma 5.5 *The barycentric subdivision* $\mathrm{Sd} = (\mathrm{Sd}_n)_{n\in\mathbb{Z}}$ *is a natural chain map* $\widetilde{S}_* \longrightarrow \widetilde{S}_*$ *which is natural chain homotopic with the identity natural chain map* $1_* : \widetilde{S}_* \longrightarrow \widetilde{S}_*$. *Moreover, for* $X \in \mathrm{ObjDTOP}$, $z \in \widetilde{S}_n(X)$ *and* $n \geq 0$, *we have* $(\mathrm{Sd}z)_\sharp \epsilon_0^n = \mathrm{Sd}\partial z$ *and* $(\mathrm{Sd}z)_\sharp \epsilon_i^n = 0$ *for* $0 < i \leq n$.

Proof. That Sd is a natural chain map $\widetilde{S}_* \longrightarrow \widetilde{S}_*$ is a simple computation using Lemma 5.1. We refer the reader to classical case in [d] Chapter III, Proposition 6.3. The fact that Sd is natural chain homotopic with the identity

natural chain map $1_* : \widetilde{S}_* \longrightarrow \widetilde{S}_*$ follows from the method of acyclic models (Theorem 2.4) since both Sd and 1_* are liftings of 1_{-1} and, by Lemma 5.3, $\widetilde{S}_*$ is acyclic and free with basis in $\mathbf{M} = \{\Delta^n : n \in \mathbb{Z}\}$.

Now let $X \in \mathrm{ObjDTOP}$, $z \in \widetilde{S}_n(X)$ and $n \geq 0$. We want to show that $(\mathrm{Sd}z)_\sharp \epsilon_0^n = \mathrm{Sd}\partial z$ and $(\mathrm{Sd}z)_\sharp \epsilon_i^n = 0$ for $0 < i \leq n$. This is clear for $n = 0$. So suppose that $n > 0$. Then $(\mathrm{Sd}z)_\sharp \epsilon_i^n = z_\sharp(b_n.\mathrm{Sd}\partial 1_{\Delta^n})_\sharp \epsilon_i^n$. The later is, by Lemma 5.1, $z_\sharp(b_n.((\mathrm{Sd}\partial 1_{\Delta^n})_\sharp \epsilon_{i-1}^{n-1}))$. Since $\partial 1_{\Delta^n} \in \widetilde{S}_{n-1}(\Delta^n)$, induction on n yields $(\mathrm{Sd}\partial 1_{\Delta^n})_\sharp \epsilon_{i-1}^{n-1} = 0$ for $i > 1$, and $(\mathrm{Sd}\partial 1_{\Delta^n})_\sharp \epsilon_0^{n-1} = \mathrm{Sd}\partial^2 1_{\Delta^n} = 0$. So $(\mathrm{Sd}z)_\sharp \epsilon_i^n = 0$ for $0 < i \leq n$. Since Sd is a natural chain map, $\mathrm{Sd}\partial z = \partial(\mathrm{Sd}z) = \sum_{i=0}^n (-1)^i (\mathrm{Sd}z)_\sharp \epsilon_i^n = (\mathrm{Sd}z)_\sharp \epsilon_0^n$. $\square$

It is at this point that the classical proof of the excision axiom for topological singular homology differs from the second author proof of the o-minimal excision axiom for the o-minimal singular homology. In the classical case we apply the Lebesgue number property to the repeated barycentric subdivision operator $\mathrm{Sd}^k = (\mathrm{Sd}_n^k)_{n\in\mathbb{Z}} : \widetilde{S}_*^{top}(X) \longrightarrow \widetilde{S}_*^{top}(X)$ where Sd^k is the composition of Sd with itself k times, to prove the following lemma (see the proof of 7.3 in [d] page 44).

Lemma 5.6 *Suppose that $X \in \mathrm{ObjTOP}$ and let U and V be open subsets of X such that $X = U \cup V$. If $z \in \widetilde{S}_n^{top}(X)$, then there is a sufficiently large $k \in \mathbb{N}$ such that $\mathrm{Sd}_n^k(z) \in \widetilde{S}_n^{top}(U) + \widetilde{S}_n^{top}(V)$.*

In the o-minimal case we use the o-minimal simplicial chain complex to define the subdivision operator which will play the role of the repeated barycentric subdivision operator.

Definition 5.7 Let K be a closed simplicial complex with a fixed total ordering of its vertices. We define the natural chain map $\tau_K : \widetilde{C}_*(K) \longrightarrow \widetilde{S}_*(|K|)$ by setting $\tau_K = 1_{\mathbb{Z}} : \widetilde{C}_{-1}(K) \longrightarrow \widetilde{S}_{-1}(|K|)$ and for $n \geq 0$, if $v_0 < \cdots < v_n$ are vertices of an n-simplex of K determining a basis element of $\widetilde{C}_*(K)$, then $\tau_K(\langle v_0, \ldots, v_n\rangle) = \sigma$ where $\sigma : \Delta^n \longrightarrow |K|$ is such that $\sigma(\sum_{i=0}^n t_i e_i) = \sum_{i=0}^n t_i v_i$.

Definition 5.8 For $i \leq n$, the definable continuous map $\gamma_i^n : \Delta^n \longrightarrow \Delta^i$ is defined by

$$\gamma_i^n(\sum_{j=0}^n a_j e_j) = \sum_{j=0}^i (a_{n-i+j} + \frac{\sum_{k=0}^{n-i-1} a_k}{i+1}) e_j.$$

So γ_i^n maps Δ^n to Δ^i by sending the first $n-i$ vertices of Δ^n to the barycenter of Δ^i.

Let (Φ, K) be a definable triangulation of Δ^n compatible with the standard simplicial complex E^n determined by Δ^n. For $X \in \mathrm{ObjDTOP}$, we define the *subdivision operator* $\mathrm{Sd}_i^K : \widetilde{S}_i(X) \longrightarrow \widetilde{S}_i(X)$ where $i \leq n$ by: $\mathrm{Sd}_{-1}^K = 1_{\mathbb{Z}}$ and for $0 \leq i \leq n$,

$$\mathrm{Sd}_i^K(z) = (\mathrm{Sd}z)_\sharp(\gamma_i^n)_\sharp(\Phi^{-1})_\sharp\tau_K\Phi_\sharp\langle e_{n-i}, \ldots, e_n\rangle$$

where $\Phi_\sharp : \widetilde{C}_*(E^n) \longrightarrow \widetilde{C}_*(K)$ is a chain map induced by $\Phi : \Delta^n \longrightarrow |K|$ (Theorem 4.2).

Note that, if $s \in E^n$ is a simplex, then for each $i \in \mathbb{Z}$ we have $\Phi_\sharp(\widetilde{C}_i(\overline{s})) \subseteq \widetilde{C}_i(K_{||\overline{s}|})$ where $(\Phi_{||\overline{s}|}, K_{||\overline{s}|})$ is the definable triangulation of $|\overline{s}|$ obtained by restriction.

We now prove the analogue of Lemma 5.5 for the subdivision operator. The role of γ_i^n in the definition of the subdivision operator is rather technical and appears in the proof of the next lemma.

Lemma 5.9 *The subdivision operator* $\mathrm{Sd}^K = (\mathrm{Sd}_i^K)_{i\leq n}$ *is a natural partial chain map* $\widetilde{S}_* \longrightarrow \widetilde{S}_*$ *of order* n*, i.e.* Sd_i^K *is natural and* $\partial\mathrm{Sd}_i^K = \mathrm{Sd}_{i-1}^K\partial$ *for each* $i \leq n$*. Moreover, there is a natural partial chain homotopy between the subdivision operator* Sd^K *and the restriction* $(1_i)_{i\leq n}$ *of the identity natural chain map* $1_* : \widetilde{S}_*(X) \longrightarrow \widetilde{S}_*(X)$*.*

Proof. It is clear from the definition that Sd_i^K is natural. We need to show that $\mathrm{Sd}^K : \widetilde{S}_*(X) \longrightarrow \widetilde{S}_*(X)$ is a partial chain map of order n for every $X \in \mathrm{ObjDTOP}$.

Let $z \in \widetilde{S}_i(X)$ where $i \leq n$. The case $i \leq 0$ is trivial. So suppose that $i > 0$. Then by the definitions and the fact that we are dealing with chain maps we get

$$\partial(\mathrm{Sd}_i^K z) = \sum_{j=0}^{i}(-1)^j(\mathrm{Sd}z)_\sharp(\gamma_i^n)_\sharp(\Phi^{-1})_\sharp\tau_K\Phi_\sharp\langle e_{n-i}, \ldots, \widehat{e}_{n-i+j}, \ldots, e_n\rangle.$$

For $n > 0$ and $i \leq n$, let $\lambda_i^n : \Delta^n \longrightarrow \Delta^{n-1}$ be the definable continuous map given by $\lambda_i^n(\sum_{i=0}^n a_ie_i) = \sum_{j\neq i}(a_j + \frac{a_i}{n})e_j$. So λ_i^n maps Δ^n to Δ^{n-1} by sending the i-th vertex of Δ^n to the barycenter of Δ^{n-1}. Since the image of

$$(\Phi^{-1})_\sharp\tau_K\Phi_\sharp\langle e_{n-i}, \ldots, \widehat{e}_{n-i+j}, \ldots, e_n\rangle$$

is contained in the closed simplex spanned by $\{e_{n-i}, \ldots, \widehat{e}_{n-i+j}, \ldots, e_n\}$, after we replace in the equality above $(\gamma_i^n)_\sharp$ by $(\epsilon_j^i)_\sharp(\lambda_j^i)_\sharp(\gamma_i^n)_\sharp$ and use Lemma 5.5, we get

$$\partial(\mathrm{Sd}_i^K z) = (\mathrm{Sd}\partial z)_\sharp(\lambda_0^i)_\sharp(\gamma_i^n)_\sharp(\Phi^{-1})_\sharp\tau_K\Phi_\sharp\langle e_{n-i+1}, \ldots, e_n\rangle,$$

which equals to $(\mathrm{Sd}\partial z)_\sharp(\gamma_{i-1}^n)_\sharp(\Phi^{-1})_\sharp\tau_K\Phi_\sharp\langle e_{n-i+1}, \ldots, e_n\rangle$ since $\lambda_0^i \circ \gamma_i^n = \gamma_{i-1}^n$. Thus $\partial(\mathrm{Sd}_i^K z) = \mathrm{Sd}_{i-1}^K(\partial z)$.

To finish the proof, we will use the method of acyclic models (Theorem 2.4). Both Sd^K and $(1_i)_{i\leq n}$ are liftings of 1_{-1}. On the other hand, by Lemma 5.3, if $\mathbf{M} = \{\Delta^n : n \in \mathbb{Z}\}$, then $\widetilde{S}_*$ is acyclic and free with basis in $\mathbf{M}$. The proof of the theorem on acyclic models ([d] Chapter VI, Proposition 11.2) shows that there is a natural partial chain homotopy $T = (T_i)_{i\leq n} : \widetilde{S}_* \longrightarrow \widetilde{S}_{*+1}$ such that $\mathrm{Sd}_i^K - 1_i = \partial T_i + T_{i-1}\partial$ for $i \leq n$. $\square$

The following lemma is the o-minimal analogue of Lemma 5.6.

Lemma 5.10 *Suppose that $X \in$ ObjDTOP and let U and V be open definable subsets of X such that $X = U \cup V$. If $z \in \widetilde{S}_n(X)$, then there is a definable triangulation (Φ, K) of Δ^n compatible with E^n such that $\mathrm{Sd}_n^K(z) \in \widetilde{S}_n(U) + \widetilde{S}_n(V)$.*

Proof. Let $C = \{h^{-1}(W) : W \in \{U, V\}$ and $h : \Delta^n \longrightarrow X \in$ MorDTOP is involved in $\mathrm{Sd}_n(z)\}$. See Section 2 for the notion of involved. Let (Ψ, L) be a definable triangulation of Δ^n compatible with C and with E^n.

Note that, for each $h : \Delta^n \longrightarrow X \in$ MorDTOP involved in $\mathrm{Sd}_n(z)$, if v is a vertex of L, $W \in \{U, V\}$ and $h \circ \Psi^{-1}(v) \in W$, then $h \circ \Psi^{-1}(|\mathrm{St}v|) \subseteq W$ where $\mathrm{St}v$ is the star of v in L. In fact, by the definition of (Ψ, L), we have $\Psi^{-1}(t) \cap h^{-1}(W) = \emptyset$ or $\Psi^{-1}(t) \subseteq h^{-1}(W)$ for every simplex t of L. Furthermore, since $\Psi \circ h^{-1}(W)$ is open in $|L|$, if the vertex v is in $\Psi \circ h^{-1}(W)$, then $s \cap \Psi \circ h^{-1}(W) \neq \emptyset$ for each $s \in \mathrm{St}v$.

Let (Φ, K) be the definable triangulation of Δ^n where K is the barycentric subdivision of simplicial complex L and $\Phi : \Delta^n \longrightarrow |L|$ is the same as $\Psi : \Delta^n \longrightarrow |K|$ (this makes sense since $|L| = |K|$). Clearly, for each simplex s of K we have $|\overline{s}| \subseteq |\mathrm{St}v|$ for some vertex v of L. Furthermore, for each $h : \Delta^n \longrightarrow X \in$ MorDTOP involved in $\mathrm{Sd}_n(z)$, we have $|L| = \Psi \circ h^{-1}(U) \cup \Psi \circ h^{-1}(V)$. Hence, either $h \circ \Phi^{-1}(|\overline{s}|) \subseteq U$ or $h \circ \Phi^{-1}(|\overline{s}|) \subseteq V$.

By the definition of $\mathrm{Sd}_n^K(z)$, we see that each $\sigma : \Delta^n \longrightarrow X \in \mathrm{MorDTOP}$ involved in $\mathrm{Sd}_n^K(z)$ is a composition of the form $h \circ \Phi^{-1} \circ g$, where h is involved in $\mathrm{Sd}_n(z)$ and $g : \Delta^n \longrightarrow |\overline{s}| \in \mathrm{MorDTOP}$ is an affine map onto the geometric realization of the closure of an n-simplex s of K. Thus $\sigma(\Delta^n) = h \circ \Phi^{-1}(|\overline{s}|) \subseteq W$ for some $W \in \{U, V\}$. Hence, $\mathrm{Sd}_n^K(z) \in \widetilde{S}_n(U) + \widetilde{S}_n(V)$. $\square$

Finally, after all the above definitions and lemmas (all from [Wo]) we are ready to prove, as in [Wo], the excision axiom for the o-minimal singular homology. The argument is now similar to the corresponding proof for the excision axiom for topological singular homology: we replace the repeated barycentric subdivision operator by the subdivision operator and use Lemmas 5.9 and 5.10 instead of Lemmas 5.5 and 5.6.

Theorem 5.11 *For every* $(X, A) \in \mathrm{ObjDTOPP}$ *and every definable open subset* W *of* X *such that* $\overline{W} \subseteq \mathring{A}$*, the inclusion* $(X - W, A - W) \longrightarrow (X, A)$ *induces isomorphisms* $H_n(X - W, A - W) \longrightarrow H_n(X, A)$ *for all* $n \in \mathbb{Z}$.

Proof. It is enough to show that if $X \in \mathrm{ObjDTOP}$ and U and V are definable subsets of X such that $X = \mathring{U} \cup \mathring{V}$, then the homomorphism $\theta_n : H_n(\widetilde{S}_*(U) + \widetilde{S}_*(V)) \longrightarrow H_n(\widetilde{S}_*(X))$ induced by the inclusion $\widetilde{S}_*(U) + \widetilde{S}_*(V) \longrightarrow \widetilde{S}_*(X)$ is an isomorphism. In fact, adapting the notation of [d] page 47, we set $\widetilde{S}_*\{U, V\} = \widetilde{S}_*(U) + \widetilde{S}_*(V)$ so that $H_n(\widetilde{S}_*\{U, V\}) \simeq H_n(\widetilde{S}_*(X))$ as in [d] Chapter III, 8.1 (d). Hence we also have that the homomorphism $H_n(U, U \cap V) \longrightarrow H_n(X, V)$ induced by the inclusion $(U, U \cap V) \longrightarrow (X, V)$ is an isomorphism. From here we get the theorem by taking $V = A$ and $U = X \setminus W$.

Below we let $\widetilde{Z}_n(X) = \mathrm{Ker}\partial_n$ and $\widetilde{B}_n(X) = \mathrm{Im}\partial_{n+1}$. Take $z \in \widetilde{Z}_n(X)$. Then by Lemma 5.10, there is a definable triangulation (Φ, K) of Δ^n compatible with E^n such that $\mathrm{Sd}_n^K(z) \in \widetilde{S}_n(\mathring{U}) + \widetilde{S}_n(\mathring{V}) \subseteq \widetilde{S}_n(U) + \widetilde{S}_n(V)$. By Lemma 5.9, Sd^K is chain homotopic to the identity on $\widetilde{S}_*(X)$. Hence, $\mathrm{Sd}_n^K(z) + \widetilde{B}_n(X) = z + \widetilde{B}_n(X)$ and θ_n is surjective.

Suppose that $z_1 + z_2 + \widetilde{B}_n(U) + \widetilde{B}_n(V) \in \mathrm{Ker}\theta_n$ where $z_1 \in \widetilde{S}_n(U)$ and $z_2 \in \widetilde{S}_n(V)$. Then there exists $w \in \widetilde{S}_{n+1}(X)$ such that $\partial w = z_1 + z_2$. By Lemma 5.10, there is a definable triangulation (Φ, K) of Δ^{n+1} compatible with E^{n+1} such that $\mathrm{Sd}_{n+1}^K(w) \in \widetilde{S}_{n+1}(U) + \widetilde{S}_{n+1}(V)$. Since Sd^K is a partial chain map,

we have $\partial \mathrm{Sd}^K_{n+1}(w) = \mathrm{Sd}^K_n(\partial w) = \mathrm{Sd}^K_n(z_1 + z_2)$. By Lemma 5.9, we have $\mathrm{Sd}^K_n(z_1+z_2) = z_1+z_2+\partial T^X_n(z_1+z_2)+T^X_{n-1}\partial(z_1+z_2) = z_1+z_2+\partial T^X_n(z_1+z_2)$, since $\partial w = z_1 + z_2$. By naturality of T, we have $T^X_n z_1 \in \widetilde{S}_{n+1}(U)$ and $T^X_n z_2 \in \widetilde{S}_{n+1}(V)$ so that $\partial T^X_n z_1 \in \widetilde{B}_n(U)$ and $\partial T^X_n z_2 \in \widetilde{B}_n(V)$. Therefore, $z_1 + z_2 + \widetilde{B}_n(U) + \widetilde{B}_n(V) = \partial \mathrm{Sd}^K_{n+1}(w)$ is zero in $H_n(\widetilde{S}_*(U) + \widetilde{S}_*(V))$ and θ_n is injective. □

6 O-minimal cohomology

The results of this section are all from [e]. They are easy consequences of previous results together with classical arguments. We include the details only for completeness.

6.1 O-minimal homology with coefficients in Q

We will now construct the o-minimal singular homology with coefficients in an R-module Q starting with the o-minimal singular homology with coefficients in $\mathbb{Z}$. In a similar way one can construct the *o-minimal simplicial homology with coefficients in* Q starting from the o-minimal simplicial homology with coefficients in $\mathbb{Z}$.

Definition 6.1 Let Q be an R-module. If $(X, A) \in \mathrm{ObjDTOPP}$, then

$$(S_*(X, A; Q), \partial_* \otimes 1_Q)$$

where $S_n(X, A; Q) = S_n(X, A) \otimes Q$, is a chain complex of R-modules. If $f : (X, A) \longrightarrow (Y, B) \in \mathrm{MorDTOPP}$, then

$$f_\sharp \otimes 1_Q : S_*(X, A; Q) \longrightarrow S_*(Y, B; Q)$$

is a chain map. We call $(S_*(X, A; Q), \partial_* \otimes 1_Q)$ the *o-minimal singular chain complex with coefficients in* Q and we often denote $f_\sharp \otimes 1_Q$ by $f_\sharp$.

Theorem 6.2 *The sequence of functors defined in Definition 6.1 determines an o-minimal homology* (H_*, d_*) *for* DTOPP *with coefficients in* Q*, called the o-minimal singular homology with coefficients in* Q*.*

Proof. This is obtained from the proof of the existence of the o-minimal singular homology with coefficients in $\mathbb{Z}$ by applying the functor $- \otimes Q$. For details compare with the classical analogue of this result presented in [d] Chapter VI, (7.1) to (7.5). $\square$

There is an analogue of Theorem 6.2 for the o-minimal simplicial homology with coefficients in Q.

6.2 O-minimal cohomology

O-minimal cohomology on C, which is either DTOPP or DCTOPP, is defined in the following way. Recall from Section 3 that $G : C \longrightarrow C$ is the functor that sends $(X, A) \in \mathrm{Obj}C$ into $(A, \emptyset) \in \mathrm{Obj}C$ and sends $f : (X, A) \longrightarrow (Y, B) \in \mathrm{Mor}C$ into $f_{|A} : (A, \emptyset) \longrightarrow (B, \emptyset) \in \mathrm{Mor}C$.

Definition 6.3 An *o-minimal cohomology* (H^*, d^*) *on* C is a sequence $(H^n)_{n\in\mathbb{Z}}$ of contravariant functors from C into the category of R-modules together with a sequence $(d^n)_{n\in\mathbb{Z}}$ of natural transformations $d^n : H^n \circ G \longrightarrow H^{n+1}$, such that the following axioms hold.

Homotopy Axiom. If $f, g : (X, A) \longrightarrow (Y, B) \in \mathrm{Mor}C$ and there is a definable homotopy in C between f and g, then

$$H^n(f) = H^n(g) : H^n(Y, B) \longrightarrow H^n(X, A)$$

for all $n \in \mathbb{Z}$.

Exactness Axiom. For $(X, A) \in \mathrm{Obj}C$, if $i : (A, \emptyset) \longrightarrow (X, \emptyset)$ and $j : (X, \emptyset) \longrightarrow (X, A)$ are the inclusions, then there exits a natural exact sequence

$$\cdots \longrightarrow H^n(X, A) \stackrel{H^n(j)}{\rightarrow} H^n(X, \emptyset) \stackrel{H^n(i)}{\rightarrow} H^n(A, \emptyset) \stackrel{d^n}{\rightarrow} H^{n+1}(X, A) \longrightarrow \cdots$$

(where the d^n's are induced by the natural transformations above).

Excision Axiom. For every $(X, A) \in \mathrm{Obj}C$ and every definable open subset U of X such that $\overline{U} \subseteq \mathring{A}$ and $(X - U, A - U) \in \mathrm{Obj}C$, the inclusion $(X - U, A - U) \longrightarrow (X, A)$ induces isomorphisms

$$H^n(X, A) \longrightarrow H^n(X - U, A - U)$$

for all $n \in \mathbb{Z}$.

Dimension Axiom. If X is a one point set, then $H^n(X, \emptyset) = 0$ for all $n \neq 0$. The R-module $H^0(X, \emptyset)$, is called the coefficient R-module.

If Q is the coefficient R-module of (H^*, d^*), we will write $H^n(X, A; Q)$ instead of just $H^n(X, A)$ and the notation $H^n(X, A)$ will stand for $H^n(X, A; \mathbb{Z})$.

Two o-minimal cohomologies (H^*, d^*) and (H'^*, d'^*) on C with coefficient R-module Q are *isomorphic* if there is a sequence $(\tau^n)_{n\in\mathbb{Z}}$ of natural equivalences $\tau^n : H^n \longrightarrow H'^n$ such that

$$\begin{array}{ccc} H^n(A;Q) & \xrightarrow{d^n} & H^{n+1}(X,A;Q) \\ \downarrow^{\tau^n} & & \downarrow^{\tau^{n+1}} \\ H'^n(A;Q) & \xrightarrow{d'^n} & H'^{n+1}(X,A;Q) \end{array}$$

commutes for all $(X, A) \in \mathrm{Obj}C$ and for all $n \in \mathbb{Z}$.

We will now construct the o-minimal singular cohomology with coefficients in an R-module Q starting with the o-minimal singular homology with coefficients in $\mathbb{Z}$. In a similar way one can construct the *o-minimal simplicial cohomology with coefficients in* Q starting from the o-minimal simplicial homology with coefficients in $\mathbb{Z}$.

Definition 6.4 Let Q is an R-module. If $(X, A) \in \mathrm{Obj}\mathrm{DTOPP}$, then we have a cochain complex of R-modules

$$(S^*(X, A; Q), \partial^*_Q)$$

where $S^n(X, A; Q)= \mathrm{Hom}(S_n(X, A), Q)$ and $\partial^n_Q = \mathrm{Hom}(\partial_{n+1}, Q)$. If $f : (X, A) \longrightarrow (Y, B) \in \mathrm{Mor}\mathrm{DTOPP}$, then

$$f^\sharp : S^*(Y, B; Q) \longrightarrow S^*(X, A; Q)$$

where $f^\sharp = \mathrm{Hom}(f_\sharp, Q)$, is a cochain map. We call $(S^*(X, A; Q), \partial^*_Q)$ the *o-minimal singular cochain complex with coefficients in* Q.

Theorem 6.5 *The sequence of functors defined in Definition 6.4 determines an o-minimal cohomology* (H^*, d^*) *in* DTOPP *with coefficients in* Q*, called the o-minimal singular cohomology with coefficients in* Q*.*

Proof. This is obtained from the proof of the existence of the o-minimal singular homology with coefficients in $\mathbb{Z}$ by applying the functor $\mathrm{Hom}(-, Q)$. For details compare with the classical analogue of this result presented in [d] Chapter VI, (7.1) to (7.5). □

There is an analogue of Theorem 6.5 for the o-minimal simplicial cohomology with coefficients in Q.

7 Comparison theorems

Here we will prove the homology version of the comparison results presented in the introduction. The cohomology version is proved dually. We will require the following lemma.

Lemma 7.1 *Let K be a simplicial complex, not necessarily closed, and let L denote the maximal closed subcomplex of the barycentric subdivision* $\mathrm{Sd}K$ *of K. Then* $\mathrm{Sd}K$ *is the star of L in* $\mathrm{Sd}K$*, and there is a definable strong deformation retraction from $|K|$ to $|L|$.*

Proof. First, note that

$$\mathrm{Sd}K = \{(b(\sigma_0), \ldots, b(\sigma_n)) : \sigma_n \in K \text{ and } \sigma_0 < \cdots < \sigma_n\},$$
$$L = \{(b(\sigma_0), \ldots, b(\sigma_n)) : \{\sigma_0, \ldots, \sigma_n\} \subseteq K \text{ and } \sigma_0 < \cdots < \sigma_n\},$$

where $b(\sigma)$ denotes the barycenter of σ. Since $(b(\sigma)) \in L$ for each $\sigma \in K$, every simplex of $\mathrm{Sd}K$ is in the star of L.

Define $r : |K| \longrightarrow |L|$ by

$$r(x) \;=\; \sum_{v \in \mathrm{Vert}L} \lambda_v(x)\, v \Big/ \sum_{v \in \mathrm{Vert}L} \lambda_v(x)$$

where λ_v is the barycentric coordinate function with respect to v. Then r is a retraction, and, since for each $x \in |K|$ the open line segment $(x, r(x))$ lies entirely within the simplex of K containing x, we have a strong deformation retraction from $|K|$ to $|L|$. $\square$

Proof of Theorem 1.2: Let (H_*, d_*) and (H', d'_*) be two o-minimal homology theories in DTOPP with coefficients in Q.

Let $(X, A) \in \mathrm{Obj}\mathrm{DTOPP}$. Take a definable triangulation (Φ, R_1) of X compatible with A and let R_2 be the barycentric subdivision of R_1 and S_2 the subcomplex of R_2 such that $\Phi(A) = |S_2|$. Let K and L be maximal closed subcomplexes of R_2 and S_2 respectively. Then $L \subseteq K$, and by Lemma 7.1 there are definable strong deformation retractions from X to $\Phi^{-1}(|K|)$ and A to $\Phi^{-1}(|L|)$.

Let $j_K : |K| \longrightarrow X$, $j_L : |L| \longrightarrow A$ and $j_{K,L} : (|K|, |L|) \longrightarrow (X, A)$ be the restrictions of Φ^{-1}. By the exactness axiom for o-minimal homology and arguing as in [dp] Theorem 4.3.1 and Corollary 4.3.2 we see that

$H_*(j_K), H_*(j_L), H'_*(j_K)$ and $H'_*(j_L)$ are isomorphisms. By the exactness axiom and the five lemma ([d] Chapter I (2.9)), it follows that $H_*(j_{K,L})$ and $H'_*(j_{K,L})$ are also isomorphisms.

Define $\tau_{X,A} : H_*(X, A; Q) \longrightarrow H'_*(X, A; Q)$ by $\tau_{X,A} = H'_*(j_{K,L}) \circ \lambda_{|K|,|L|} \circ (H_*(j_{K,L}))^{-1}$ where λ is the isomorphism between (H_*, d_*) and (H', d'_*) when restricted to DCTOPP. By construction and the fact that $d'_* \circ \lambda_{|K|,|L|} = \lambda_{|L|} \circ d_*$ we have $d'_* \circ \tau_{X,A} = \tau_A \circ d_*$.

We show that $\tau_{Y,B} \circ H_*(f) = H'_*(f) \circ \tau_{X,A}$ for every $f : (X, A) \longrightarrow (Y, B)$ in MorDTOPP. Let $(K_1, L_1) \in$ ObjCSCP as required for the definition of $\tau_{X,A}$. Let (Φ, R_1) be a definable triangulation of Y compatible with B, $f(j_{K_1}(|K_1|))$ and $f(j_{L_1}(|L_1|))$, and let R_2 be the barycentric subdivision of R_1 and S_2 the subcomplex of R_2 such that $\phi(B) = |S_2|$. Let K and L be maximal closed subcomplexes of R_2 and S_2 respectively. Then $L \subseteq K$, and there are definable strong deformation retractions from Y to $\Phi^{-1}(|K|)$ and B to $\Phi^{-1}(|L|)$, so (K, L) are as required for the definition of $\tau_{Y,B}$. By [vdd] Chapter VI, Proposition 1.10, $f(j_{K_1}(|K_1|))$ and $f(j_{L_1}(|L_1|))$ are both closed and bounded definable sets. It follows that $f(j_{K_1}(|K_1|)$ (resp., $f(j_{L_1}(|L_1|))$ is a subset of $j_K(|K|)$ (resp., $j_L(|L|)$, and hence we have definable maps $f^K_{K_1} = (j_{K|f(j_{K_1}(|K_1|))})^{-1} \circ f_{|j_{K_1}(|K_1|)} \circ j_{K_1} : |K_1| \longrightarrow |K|$, $f^L_{L_1} = (j_{L|f(j_{L_1}(|L_1|))})^{-1} \circ f_{|j_{L_1}(|L_1|)} \circ j_{L_1} : |L_1| \longrightarrow |L|$ and $f^{K,L}_{K_1,L_1} : (|K_1|, |L_1|) \longrightarrow (|K|, |L|)$. Note that $f \circ j_{K_1,L_1} = j_{K,L} \circ f^{K,L}_{K_1,L_1}$.

Now using the definition of $\tau_{X,A}$ and $\tau_{Y,B}$ and the fact that $\lambda_{|K|,|L|} \circ H_*(f^{K,L}_{K_1,L_1}) = H'_*(f^{K,L}_{K_1,L_1}) \circ \lambda_{|K_1|,|L_1|}$ it follows that $\tau_{Y,B} \circ H_*(f) = H'_*(f) \circ \tau_{X,A}$ as required.

Now that naturality of τ has been established, we can see that τ is independent of the choice of triangulation by considering naturality with respect to the identity $1_{(X,A)}$. $\square$

We will now give a proof of the homology version of Theorem 1.4 (the cohomology version is proved dually). This proof closely follows the proof of Theorem 1.2.

Let FTTOPP denote the category of pairs of topological spaces which are finitely triangulable over $\mathbb{R}$. The simplicial complexes in the triangulations are not required to be closed.

Before we begin the proof itself, we define a map of a simplicial complex onto itself which, under the right circumstances, retracts a neighborhood of

a subcomplex onto the subcomplex. Let K be a simplicial complex, $L \subseteq K$, $V \subseteq \mathrm{Vert}K$, and $\overline{V} = \mathrm{Vert}K - V$. Let $b(\sigma)$ denote the barycenter of σ. We define the span of V in K, denoted $\mathrm{Sp}_K V$, the star of L in K, denoted $\mathrm{St}_K L$, and the barycentric subdivision of K, denoted $\mathrm{Sd}K$, by

$$\begin{aligned}
&\mathrm{Sp}_K V = \{\, \sigma \in K : \mathrm{Vert}(\sigma) \subseteq V \,\}, \\
&\mathrm{St}_K L = \{\, \sigma \in K : \exists \tau (\tau < \sigma \cap \tau \in L) \,\}, \\
&\mathrm{Sd}K = \{\, (b(\sigma_0), \ldots, b(\sigma_n)) : \sigma_n \in K \text{ and } \sigma_0 < \cdots < \sigma_n \,\}.
\end{aligned}$$

Note that $\mathrm{Sp}_{\mathrm{Sd}K}\{\, b(\sigma) : \sigma \in L \,\}$ is the maximal closed subcomplex of $\mathrm{Sd}L$.

Define the function $\alpha_V : |K| \longrightarrow [0,1]$ by

$$\alpha_V(x) = \sum_{v \in V} \lambda_v(x),$$

where λ_v is the barycentric coordinate function with respect to v. Note that $\alpha_V^{-1}(1) = |\mathrm{Sp}_K V|$ and $\alpha_V^{-1}((0,1]) = |\mathrm{St}_K V|$. Also, $\alpha_{\overline{V}}(x) = 1 - \alpha_V(x)$.

Define the function $r_V : |\mathrm{St}_K V| \longrightarrow |\mathrm{Sp}_K V|$ by

$$r_V(x) = \frac{1}{\alpha_V(x)} \sum_{v \in V} \lambda_v(x)\, v.$$

Note that $(1 - \alpha_V(x)) r_{\overline{V}}(x) + \alpha_V(x) r_V(x) = x$.

Let $\epsilon \in (0,1)$. Define $s_{\epsilon,V} : |K| \longrightarrow |K|$ by

$$s_{\epsilon,V}(x) = \begin{cases} x & \text{if } \alpha_V(x) = 0, \\ r_V(x) & \text{if } \alpha_V(x) \geq \epsilon, \\ \dfrac{\epsilon - \alpha_V(x)}{\epsilon}\, r_{\overline{V}}(x) + \dfrac{\alpha_V(x)}{\epsilon}\, r_V(x) & \text{if } \alpha_V(x) \in (0,\epsilon). \end{cases}$$

Then $s_{\epsilon,V}(\alpha_V^{-1}([\epsilon,1])) \subseteq |\mathrm{Sp}_K V|$ and $s_{\epsilon,V} \simeq 1_{|K|}$. With the function $s_{\epsilon,V}$ available, we can proceed with the proof of the theorem.

Proof of Theorem 1.4: Let (H_*, d_*) and (H'_*, d'_*) be two homology theories on FTTOPP. Let $(X, A) \in \mathrm{Obj}\mathrm{FTTOPP}$. By definition, there exists a triangulation (Φ, R) of X compatible with A. As in the proof of Theorem 1.2, by taking maximal closed subcomplexes of the barycentric subdivision of

R, we obtain closed simplicial complexes K and K', with $|K'| \subseteq |K|$, which allow us to define the map $\tau_{X,A}$, and $d'_* \circ \tau_{X,A} = \tau_A \circ d_*$.

Let $f : (X, A) \longrightarrow (Y, B) \in$ MorFTTOPP. We wish to show that $\tau_{Y,B} \circ H_*(f) = H'_*(f) \circ \tau_{X,A}$.

There exists a triangulation (Ψ, S) of Y compatible with B. Let S' be the subcomplex such that $|S'| = \Psi(B)$, and let L and L' be maximal closed subcomplexes of the barycentric subdivisions of S and S'. Then $f(|K'|) \subseteq S' \subseteq \mathrm{St}_{\mathrm{Sd}S} L'$. Let $V = \mathrm{Vert} L'$. Since $f(|K'|)$ is compact and $f(|K'|) \subseteq \alpha_V^{-1}((0, 1])$, there exists $\epsilon \in (0, 1)$ such that $f(|K'|) \subseteq \alpha_V^{-1}([\epsilon, 1])$. Then $s_{\epsilon,V} \circ f(|K'|) \subseteq |L'|$. Now let

$$g = r_{\mathrm{Vert}L} \circ s_{\epsilon,V} \circ f \circ j_{K,K'}.$$

Then g provides a map $(|K|, |K'|) \longrightarrow (|L|, |L'|)$ such that $j_{L,L'} \circ g \simeq f \circ j_{K,K'}$. Now using the definition of $\tau_{X,A}$ and $\tau_{Y,B}$ and the fact that $\lambda_{|L|,|L'|} \circ H_*(g) = H'_*(g) \circ \lambda_{|K|,|K'|}$, it follows that $\tau_{Y,B} \circ H_*(f) = H'_*(f) \circ \tau_{X,A}$.

Now that naturality of τ has been established, we can see that τ is independent of the choice of triangulation by considering naturality with respect to the identity $1_{(X,A)}$. $\square$

References

[bo] A.Berarducci and M.Otero *O-minimal fundamental group, homology and manifolds* J. London Math. Soc. 65 (2) (2002) 257-270.

[BCR] J.Bochnak, M. Coste and M-F. Roy *Real algebraic geometry* Springer-Verlag 1998.

[D1] H.Delfs *Kohomologie affiner semialgebraischer Räume* Diss. Univ. Regensburg 1980.

[D2] H.Delfs *Homology of locally semialgebraic spaces* Lecture Notes in Mathematics 1484 Springer-Verlag 1991.

[dk] H.Delfs and M.Knebusch *On the homology of algebraic varieties over real closed fields* J. reine u.angew. Math. 335 (1982) 122-163.

[dp] C.Dodson and P.Parker *A user's guide to algebraic topology* Kluwer Academic Publishers 1997.

[d] A.Dold *Lectures on algebraic topology* Springer Verlag 1980.

[vdd] L. van den Dries *Tame topology and o-minimal structures* Cambridge University Press 1998.

[dd] J.Denef and L.van den Dries *p-adic and real subanalytic sets* Ann. Math. 128 (1988) 79-138.

[dm] L. van den Dries and C.Miller *On the real exponential field with restricted analytic functions* Israel J. Math. 85 (1994) 19-56.

[ds1] L. van den Dries and P. Speissegger *The real field with convergent generalized power series* Trans. Amer. Math. Soc. 350 (11) (1998) 4377-4421.

[ds2] L. van den Dries and P. Speissegger *The field of reals with multisummable series and the exponential function* Proc. London Math. Soc. 81 (3) (2000) 513-565.

[e] M.Edmundo *Invariance of o-minimal singular (co)homology in elementary extensions* Séminaire de Structure Algébriques Ordennées n. 76 (2004) (ed., F. Delon et al.) Publications de la Université de Paris VII.

[es] S.Eilenberg and N.Steenrod *Foundations of algebraic topology* Princeton University Press 1952.

[k] M.Knebusch *Semi-algebraic topology in the last ten years* in Real Algebraic Geometry, Lecture Notes in Mathematics 1524, Springer-Verlag 1992.

[KPS] J.Knight, A.Pillay and C. Steinhorn *Definable sets in ordered structures II* Trans. Amer. Math. Soc. 295 (1986) 593-605.

[ps] Y.Peterzil and C.Steinhorn *Definable compacteness and definable subgroups of o-minimal groups* J. London Math. Soc. 59 (2) (1999) 769–786.

[Sh] S.Shelah *Classification theory and the number of non isomorphic models* 2nd ed. revised, North-Holland 1990.

[sp] E. H. Spanier *Algebraic topology* McGraw-Hill 1966.

[Wo] A.Woerheide *O-minimal homology* PhD. Thesis (1996), University of Illinois at Urbana-Champaign.

[w] A.Wilkie *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function* J. Amer. Math. Soc. 9 (1996) 1051-1094.