Divisors in Global Analytic Sets

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Abstract. We prove that any divisor Y of a global analytic set $X \subset \mathbb{R}^n$ has a generic equation, that is, there is an analytic function vanishing on Y with multiplicity one along each irreducible component of Y. We also prove that there are functions with arbitrary multiplicities along Y. The main result states that if X is coherent, Y is locally principal, $X \setminus Y$ is not connected and Y represents the zero class in $\mathrm{H}^{\infty}_{q-1}(X, \mathbb{Z}_2)$ then the divisor Y is globally principal.

Keywords: Real analytic sets, divisors.

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Introduction

In this paper we prove that any divisor Y of a global analytic set $X \subset \mathbb{R}^n$ has a generic equation, that is, there is an analytic function vanishing on Y with multiplicity one along each irreducible component of Y (we refer to section 2 below for the definition of divisor). Furthermore, it is proved that there are functions with arbitrary multiplicities along Y. Unfortunately we cannot get, in general, that Y is the zero set of this equation. Thus, one can ask under what conditions there is a global analytic function g such that $Y = \operatorname{div}(g)$, in other words g generates the ideal $\mathcal{I}_Y$. We find, at least when the ambient space X is coherent, two types of conditions. The first one is an obvious local condition; the divisor must be *locally principal*. It is easy to find examples where this fact does not occur, even when the divisor has codimension 1 at any point. The second condition is a topological condition: a principal divisor Y has always null fundamental class in the group $\mathrm{H}^{\infty}_{q-1}(X, \mathbb{Z}_2)$. Also a third topological condition is required, that is $X \setminus Y$ is not connected. This is because a generator of $\mathcal{I}_Y$ cannot have constant sign.

We are able to prove that these conditions are sufficient for Y to be principal only when X is coherent; in this case we have Cartan's theorem B and an isomorphism between the groups $H^1(X, \mathcal{O}^*)$ and $H^1(X, \mathbb{Z}/2)$. So, the conditions above imply that the line bundle defined by the local generators of $\mathcal{I}_Y$ is trivial what, in turn, makes us able to find a global generator, see Theorem 3.3 below. As far as we know, this result was known only for analytic manifolds.

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Putting together the discussion about multiplicities along a divisor with the principality conditions, we prove an equivalent condition to an ideal of the type

$$\prod_i \mathfrak{p}_i{}^{a_i}$$

be principal, where $\mathfrak{p}_i$ is a prime ideal in $\mathcal{O}(X)$, namely that the associated divisor $Y = \sum a_i Y_i$, where Y_i is the zero set of $\mathfrak{p}_i$, has vanishing fundamental class $[Y] = 0 \in H^{\infty}_{q-i}(X, \mathbb{Z}/2)$, its support disconnects X and for all i the ideal sheaf $\mathfrak{p}_i \mathcal{O}_X$ is locally principal.
These results can be seen as a consequence of the solution of Cousin's Second Problem (see, for example, chapter VIII of [GR65]) in the case of an analytic manifold X such that $\mathrm{H}^2(X, \mathbb{Z}) = 0$ but here we only assume that X is coherent.
The paper is organized as follows: In the first section we give some definitions and preliminary results on multiplicities. Section 2 is devoted to the problem of finding a generic equation of a divisor and a positive equation with arbitrary given multiplicities. Finally, in section 3 we prove (for a coherent ambient space X) the condition above to get $Y = \mathrm{div}(g)$.

1 Multiplicities

Let $X \subset \mathbb{R}^n$ be a global analytic set, i.e. the zero set of finitely many analytic functions in $\mathcal{O}_{\mathbb{R}^n} = \mathcal{O}_n$. Recall that a global analytic set admits coherent structures and admits complexifications, i.e. there exists a coherent ideal sheaf $\mathcal{F} \subset \mathcal{O}_n$ such that $X = \mathrm{Supp}\ \mathcal{O}_n/\mathcal{F}$ and there exists a complex analytic space $\widetilde{X}$ in a suitable complex Stein neighbourhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ such that $\widetilde{X} \cap \mathbb{R}^n = X$; moreover these three properties (being global, admitting a coherent structure and being the real part of a complex analytic set) are equivalent (see Prop. 15 in [Car57] and [Tog67]). One can prove that among these coherent sheaves there is a largest one, which we will denote by $\mathcal{I}_X$; also among these complex analytic sets there is a smallest one, say still $\widetilde{X}$. Moreover, for any real point x, $\mathcal{I}_{\widetilde{X},x} = \mathcal{I}_{X,x} \otimes_{\mathbb{R}} \mathbb{C}$, i.e. they define on X the same structure, the so called *well reduced structure* (cf. [ABT75], [Gal76]). Also it can be checked that $\mathcal{I}_X$ is precisely the sheaf generated by the ideal of analytic functions vanishing on X, namely $\mathcal{I}_{X,x} = I(X)\mathcal{O}_{n,x}$ where $I(X) = \{f \in \mathcal{O}(\mathbb{R}^n) \mid f = 0 \text{ on } X\}$ (cf. [BP]).
We shall call $\mathcal{O}_X = \mathcal{O}_n/\mathcal{I}_X$ the sheaf of analytic functions on X. The ring of global sections of this sheaf is $\mathcal{O}(X) = \mathcal{O}(\mathbb{R}^n)/I(X)$ and the ring $\mathcal{M}(X)$ of meromorphic functions on X will be defined as the total ring of fractions of $\mathcal{O}(X)$.
Remark that the ideal $I(X)$ may be prime even if X is not irreducible as analytic space. A classical example is the following

$$X = \{x^2 - (z^2-1)y^2 = 0\} \subset \mathbb{R}^3$$

The polynomial $p = x^2 - (z^2-1)y^2$ is irreducible as analytic function and generates $I(X)$. Nevertheless, X is the union of two analytic subspaces X_1 and X_2 that are not global, each one isomorphic to a Whitney umbrella.

So, from now on we shall call a global analytic set X *irreducible* if it does not admit proper global analytic subsets of the same dimension, that is, if $I(X)$ is a prime ideal in $\mathcal{O}(\mathbb{R}^n)$.

Now, let $Y \subset X$ be an irreducible global analytic subset of codimension 1. We define the coherent sheaf of ideals $\mathcal{I}_Y$ as $\mathcal{I}_{Y,x} = I(Y)\mathcal{O}_{X,x}$, where $I(Y) = \{f \in \mathcal{O}(X) \,|\, f = 0 \text{ on } Y\}$.
Suppose that at some point $x \in Y$ the ideal $\mathcal{I}_{Y,x}$ is principal, say, $\mathcal{I}_{Y,x} = g\mathcal{O}_{X,x}$ for some $g \in \mathcal{O}_{X,x}$. Then, the germ of any $f \in \mathcal{O}(X)$ at x can be written as $f_x = g^r v$ for some non-negative integer r and some $v \notin \mathcal{I}_{Y,x}$. The integer r will be called the *multiplicity* of f along Y at the point x and it will be denoted as $m_{Y,x}(f)$. The multiplicity of a meromorphic function $f = \dfrac{f_1}{f_2} \in \mathcal{M}(X)$ where $f_1, f_2 \in \mathcal{O}(X)$ (and f_2 is not a zero divisor of $\mathcal{O}(X)$) is defined as $m_{Y,x}(f) = m_{Y,x}(f_1) - m_{Y,x}(f_2)$. It is straightforward to check that

$$V_{Y,x} := \{f \in \mathcal{M}(X) \,|\, m_{Y,x}(f) \geq 0\} \supset \mathcal{O}(X)$$

is a discrete valuation ring.
Next lemma assures that $m_{Y,x}$ and, consequently, also $V_{Y,x}$ do not depend on the point $x \in Y$ at which $\mathcal{I}_{Y,x}$ is principal. Thus we will just write m_Y and V_Y.

Lemma 1.1. *Let $f \in \mathcal{O}(X)$, let $Y \subset X$ be an irreducible global analytic subset of codimension 1 and let r be a non-negative integer. Then the following conditions are equivalent:*

- *a)* $f \in I(Y)^r$.
- *b)* $f_p \in \mathcal{I}^r_{Y,p}$ *for some* $p \in Y$.
- *c)* $f_p \in \mathcal{I}^r_{Y,p}$ *for all* $p \in Y$.

Proof: a) $\Rightarrow$ c) If $f \in I(Y)^r$ then by definition of the ideal $\mathcal{I}_{Y,x}$ it is $f_x \in \mathcal{I}^r_{Y,x}$.
c) $\Rightarrow$ b) Trivial.
b) $\Rightarrow$ a) Suppose that $f_p \in \mathcal{I}^r_{Y,p}$ for some $p \in Y$. We define the ideal sheaf $\mathcal{J} = \mathcal{I}^r_Y : (f)$ whose stalk at $x \in X$ is $\mathcal{J}_x = \{g \in \mathcal{O}_{X,x} \,|\, gf_x \in \mathcal{I}^r_{Y,x}\}$. The ideal sheaf $\mathcal{J}$ is coherent. Indeed, since the ideal sheaf $\mathcal{I}^r_Y \cap (f)$ is coherent, if $g_1, \ldots, g_q$ are generators of its stalk at x, they generate $\mathcal{I}^r_Y \cap (f)$ in a neighbourhood of x. So, $g_1/f, \ldots, g_q/f$ generate $\mathcal{J}_y$ for y in a neighbourhood of x and $\mathcal{J}$ is coherent.
By Cartan's Theorem A the ideal $\mathcal{J}_p$ is generated by global sections of $\mathcal{J}$; by hypothesis $\mathcal{J}_p = \mathcal{O}_{X,p}$, so, in particular, there is a global section h of $\mathcal{J}$ such that $h(p) \neq 0$. Of course, we have $h \in \mathcal{O}(X)$.
By construction, $fh \in I(Y)^r$ (note that $\mathcal{I}^r_{Y,x} = I(Y)^r\mathcal{O}_{X,x}$ for every $x \in X$). Now, as $I(Y)^r$ is a primary ideal, if $f \notin I(Y)^r$ then, it would be $h^s \in I(Y)^r$ for some integer $s > 0$ which is not because $h(p) \neq 0$. Thus, $f \in I(Y)^r$ and we are done.
Finally, from these equivalences if $m_{Y,x}(f) = r$ then $f_x \in \mathcal{I}^r_{Y,x} \setminus \mathcal{I}^{r+1}_{Y,x}$ and so, $f \in I(Y)^r \setminus I(Y)^{r+1}$. Hence $m_{Y,x}(f) = \max\{s \in \mathbb{N} \,|\, f \in I(Y)^s\}$. □

In particular, the valuation m_Y can be defined if $Y \cap \operatorname{Reg} X \neq \emptyset$. For take some point $x \in \operatorname{Reg} X$ where Y has maximal dimension: then $\mathcal{O}_{X,x}$ is a unique factorization domain and $\mathcal{I}_{Y,x}$ is principal. As a consequence, in this case we have

$$m_Y(f) = \max\{s \in \mathbb{N} \,|\, f \in I(Y)^s\}.$$

Next proposition gives another characterization of V_Y.

Proposition 1.2. *Let $Y \subset X$ be an irreducible global analytic subset of codimension 1 such that $Y \cap \operatorname{Reg} X \neq \emptyset$. Then $V_Y = \mathcal{O}(X)_{I(Y)}$.*
In particular, m_Y is a real valuation.

Proof: First of all, it is easy to check that $V_Y \supset \mathcal{O}(X)_{I(Y)}$.
To prove the other inclusion, take some point $x \in \operatorname{Reg} Y \cap \operatorname{Reg} X$, which exists, since otherwise, $\operatorname{Reg} Y \subset \operatorname{Sing} X$ and then, $Y \subset \operatorname{Sing} X$. Let $\mathfrak{m}_x \subset \mathcal{O}(X)$ be the ideal of analytic functions on X vanishing at x. As $\mathfrak{m}_x \supset I(Y)$, we have that $\mathcal{O}(X)_{I(Y)} = (\mathcal{O}(X)_{\mathfrak{m}_x})_{I(Y)}$. The ring $\mathcal{O}(X)_{\mathfrak{m}_x}$ is regular, cf. [ABR96], Proposition VIII.4.4 , so, its localization at $I(Y)\mathcal{O}(X)_{\mathfrak{m}_x}$, which is a prime ideal of height one, is a discrete valuation ring. Hence, $\mathcal{O}(X)_{I(Y)} \supset V_Y$.
Finally, note that the residue field of $\mathcal{O}(X)_{I(Y)}$ is the field of meromorphic functions on Y which is a real field. □

Next, we are to see that given Y as above we can find a uniformizer $h \in \mathcal{O}(X)$ of m_Y generating $\mathcal{I}_{Y,x}$ for almost all points $x \in Y$. We recall that a global analytic subset $W \subset X$ always admits a *positive equation*, that is a non-negative function $g \in \mathcal{O}(X)$ whose zero set is $\mathcal{Z}(g) = W$. One can take, for instance, $g = f_1^2 + \cdots + f_q^2$, where $f_1, \ldots, f_q \in \mathcal{O}(X)$ are such that $W = \{f_1 = 0, \ldots, f_q = 0\}$. Note that any such equation has multiplicity greater than 1 over Y.

Lemma 1.3. *Let $Y \subset X$ be an irreducible global analytic subset of codimension 1 such that $\mathcal{I}_{Y,p}$ is principal for some $p \in Y$. Then there is a uniformizer $h \in \mathcal{O}(X)$ of m_Y such that $h_x\mathcal{O}_{X,x} = \mathcal{I}_{Y,x}$ for all $x \in Y$ up to a real analytic set of codimension 1 in Y. Moreover, given any global analytic subset $Y' \subset X$ such that $Y \not\subset Y'$ the uniformizer h can be chosen so that $\mathcal{Z}(h) \cap Y'$ has no components of codimension 1 in X.*

Proof: By Cartan's Theorem A there is a finite number of global analytic functions on X which generate the ideal $\mathcal{I}_{Y,p}$. At least one of these functions, call it f, has multiplicity one at p.
Let $\widetilde{X}, \widetilde{Y} \subset \Omega \subset \mathbb{C}^n$, where Ω is a Stein open neighbourhood of $\mathbb{R}^n$ in $\mathbb{C}^n$, be complexifications of X and Y, respectively. Up to shrink Ω, the function $f \in \mathcal{O}(X)$ can be extended to a global analytic function on $\widetilde{X}$, which will be still called f. The ideal $\mathcal{I}_{\widetilde{Y},p} = I(\widetilde{Y})\mathcal{O}_{\widetilde{X},p}$ is also principal and $f_p = v_p g$, where $v_p \in \mathcal{O}_{\widetilde{X},p} \setminus \mathcal{I}_{\widetilde{Y},p}$ and $g \in \mathcal{O}_{\widetilde{X},p}$ is a generator of $\mathcal{I}_{\widetilde{Y},p}$. Then, in a small complex neighbourhood U of p where v is defined, f_x generates $\mathcal{I}_{\widetilde{Y},x}$ for all $x \in \widetilde{Y} \cap U \setminus \{v = 0\}$. This last set

is not empty, because $\widetilde{Y}$ is pure dimensional; so, the set of points at which f_x is a generator of $\mathcal{I}_{\widetilde{Y},x}$ is not empty.

Consider the coherent sheaf of ideals $\mathcal{J}$ defined by $\mathcal{J}_x = (f_x \mathcal{O}_{\widetilde{X},x} : \mathcal{I}_{\widetilde{Y},x})$, where $x \in \widetilde{X}$, that is, $h_x \in \mathcal{J}_x$ if and only if $h_x \mathcal{I}_{\widetilde{Y},x} \subset f_x \mathcal{O}_{\widetilde{X},x}$. Thus $\mathcal{I}_x = \mathcal{O}_{\widetilde{X},x}$ if and only if f_x generates $\mathcal{I}_{\widetilde{Y},x}$. Therefore, the support

$$\operatorname{supp}(\mathcal{O}_{\widetilde{X}}/\mathcal{J}) = \{x \in \widetilde{X} \mid f_x \text{ does not generates } \mathcal{I}_{\widetilde{Y},x}\}$$

is a closed analytic set $\widetilde{W}$ which does not contain $\widetilde{Y}$. As $\widetilde{Y}$ is irreducible, $\widetilde{Y} \cap \widetilde{W}$ has at least codimension 1 in $\widetilde{Y}$. Hence, f_x generates $\mathcal{I}_{\widetilde{Y},x}$ for all $x \in \widetilde{Y} \setminus \widetilde{W}$. Then, also f_x generates $\mathcal{I}_{Y,x}$ for all $x \in Y \setminus W$, where $W = \widetilde{W} \cap \mathbb{R}^n$.

Note that $W \cap Y$ is a subset of codimension at least 1 in Y. For suppose that $W \supset Y_{\max}$, where $Y_{\max}$ denotes the part of maximal dimension of Y. Then it would be $W \supset Y$ and so $\widetilde{W} \supset Y$. But as $\widetilde{Y}$ is the complexification of Y this in turn would imply $\widetilde{W} \supset \widetilde{Y}$, which is a contradiction.

Now, let Y' be any analytic set not containing Y. Take positive equations $f_Y, f_{Y'} \in \mathcal{O}(X)$ of Y and Y', respectively. Then $\overline{f} = f_{Y'} f + f_Y$ has the required properties.

□

2 Divisors

Let X be a global analytic set in $\mathbb{R}^n$ as before. Set $q = \dim X$

Definition 2.1. *Let $\{Y_i\}, i \in J$ be a locally finite family of global irreducible analytic subsets of X, with for any i $\dim Y_i = q-1$ and $Y_i \cap Reg X \neq \emptyset$. We shall call a* divisor *in X the following sum:*

$$\sum_{i \in J} n_i Y_i$$

where $n_i \in \mathbb{Z}$. The divisor is called reduced *if for all i $n_i = 1$. The* support *of a divisor is the global analytic set $Y = \bigcup_i Y_i$. It is a global analytic subset of X, because the family $\{Y_i\}, i \in J$ is locally finite.*

Finally we say that two divisors Y, Y' are coprime *if their supports do not share any irreducible component.*

The set $\mathcal{D}$ of divisors has a natural structure of abelian group.

For the components of a divisor multiplicities m_{Y_i} are well defined. We shall say that $Y = \sum_{i \in J} n_i Y_i$ is the divisor of an analytic function g and we shall write $Y = \operatorname{div}(g)$ if $m_{Y_i}(g) = n_i$ and the zero set of g is the support of Y. In this case we shall call Y *principal.*

Let Y be (the support of) a divisor. Now by classical results on triangulations (cf. [Łoj64]), we may find a locally finite triangulation of the couple (X, Y); this means that we have a simplicial complex K, together with a subcomplex K_Y, and

a homeomorphism $f : K \to X$ such that $f(K_Y) = Y$ and for each simplex σ of K, the restriction $f|_{\mathring{\sigma}}$ is an analytic isomorphism. So, for any j, we have isomorphisms $f_* : \mathrm{H}_j^\infty(K, \mathbb{Z}_2) \to \mathrm{H}_j^\infty(X, \mathbb{Z}_2)$,
Here $\mathrm{H}_j^\infty(X, \mathbb{Z}_2)$ is the homology group based on infinite chains, for the definition and generalities on the groups $\mathrm{H}_j^\infty(X, \mathbb{Z}_2)$ we refer to [Mas78].
Also, by the construction above, each component Y_i of the divisor defines in a natural way an element $[Y_i]$ in the group $\mathrm{H}_{q-1}^\infty(X, \mathbb{Z}_2)$. Since any two such triangulations are P-L-equivalent by hauptvermutung (cf. [SY84]), this fact allows to define a group homomorphism

$$\mathcal{D} \to \mathrm{H}_{q-1}^\infty(X, \mathbb{Z}_2)$$

sending the divisor Y to the class $\sum_i n_i[Y_i]$.

¿From now on we shall use the same name for both a divisor and its support when there is no risk of confusion.

We will find for any reduced divisor Y of X what we shall call a *generic equation*, that is, we shall find an analytic function h vanishing on Y with multiplicity 1 along each component Y_i of Y.

Theorem 2.2. *(Generic equation) Let $Y = \sum Y_i$ be a reduced divisor of X. Then there exists $h \in \mathcal{O}(X)$ such that $m_{Y_i}(h) = 1$ for all $i \in I$.*
In particular, h changes sign at every point of maximal dimension of Y and it is a local generator of $\mathcal{I}_{Y,x}$ for all $x \in Y$ up to an analytic set of codimension 1 in Y.
Moreover, given any global analytic subset $W \subset X$ not containing any component of Y the function h can be chosen such that $\mathcal{Z}(h) \cap W$ has codimension at least 2 in X.

Proof: For each $x \in X$ we set J_x for the finite set of indices $i \in J$ such that $x \in Y_i$. Then we define the coherent sheaf of ideals $\mathcal{J}$ as

$$\mathcal{J}_x = \left(\prod_{i \,|\, i \in J_x} h_{i,x}, \prod_{i \,|\, i \in J_x} g_{Y_i,x} \right),$$

where each h_i is a uniformizer of m_{Y_i} not vanishing on Y_j, for $j \neq i$, cf. lemma 1.3, and g_{Y_i} is a positive equation of Y_i. Since $\mathcal{J}_x$ is generated by two functions the sheaf $\mathcal{J}$ is globally generated by finitely many global sections $f_1, \dots, f_r$, cf. [Coe67]. Note that for each Y_i at least one f_j has multiplicity one along Y_i.
Set $I_0 = \emptyset$ and define $I_j = \{i \in I \mid m_{Y_i}(f_j) = 1\} \setminus (\bigcup_{t=0}^{j-1} I_t)$.
We define the functions $f'_j = f_j + e_j$, $j = 1, \dots, r$ where e_j is a positive equation of $\bigcup_{i \in I_j} Y_i$.
Note that $\mathcal{Z}(f'_j) = \bigcup_{i \in I_j} Y_i$ and $m_{Y_i}(f'_j) = 1$ for $i \in I_j$. Moreover for each Y_i there is exactly one f'_j vanishing on Y_i. So $f = f'_1 \dots f'_r$ has multiplicity one along each Y_i.
Finally, by the same trick as in the proof of lemma 1.3, if $g_Y, g_W \in \mathcal{O}(X)$ are positive equations of Y and W, respectively, the zero set of $\overline{f} = g_W f + g_Y$ cuts W along a set of codimension at least 2. □

This theorem says, in particular, that $\mathrm{div}(h) = Y + Y'$ for some divisor Y' coprime with Y. We can say little about the divisor Y' of "extra" zeroes of the function h, except that it can be chosen coprime with any divisor W fixed in advance.
Thus, two questions arise. Is it possible to find a generic equation of Y being a local generator of $\mathfrak{I}_{Y,x}$ at every $x \in Y$? And, what can be said about Y'? In the next section we will answer these questions under some additional hypothesis on the sheaf $\mathfrak{I}_Y$ and on the ambient space X.
If Y is a divisor then, a positive equation of Y has even multiplicity along each component Y_i of Y. What we show in the next theorem is that given any sequence of even positive integers $\{m_i = 2n_i\}_{i\in I}$ we can find a positive equation of Y with precisely multiplicity m_i along each Y_i.

Theorem 2.3. *(Positive equation) Let $Y = \sum 2n_iY_i$ be a positive even divisor Then, there is a positive analytic function h such that $Y = \mathrm{div}(h)$.*

Proof: With the same notations of the previous theorem we define the coherent sheaf

$$\mathcal{J}_x = \left(\prod_{i \mid i \in J_x} h_{i,x}^{n_i}, \prod_{i \mid i \in J_x} g_{Y_i,x}^{n_i} \right).$$

Again by [Coe67] this sheaf is generated by a finite number of global sections $f_1, \dots, f_r$. Let $h = f_1^2 + \dots + f_r^2$. It is straightforward to see that h is a positive equation of $Y = \cup Y_i$.
Now, for a given Y_i take some point $x \in Y_i \setminus \bigcup_{j\neq i} Y_j$ such that $h_{i,x}$ generates $\mathfrak{I}_{Y_i,x}$. Then, $h_{i,x}^{n_i}$ generates $\mathcal{J}_x$ so, $(h_{i,x}^{n_i}) = \mathcal{J}_x = (f_{1,x}, \dots, f_{r,x})$. Thus, $m_{Y_i}(f_\ell) \geq n_i$ for all $\ell = 1, \dots, r$ and $m_{Y_i}(f_k) = n_i$ for some k. As m_{Y_i} is a real valuation, we have that $m_{Y_i}(h) = 2 \min_\ell m_{Y_i}(f_\ell) = 2n_i$. □

As a corollary of the last two theorems, we prove that for any divisor $Y = \sum_i n_iY_i$ there is a meromorphic function f such that $m_{Y_i}(g) = n_i$ for each $i \in I$. But note again that, unless all multiplicities are even, the set of points where f is zero or not analytic can be strictly larger than $\mathrm{supp}\, Y$, i.e. $\mathrm{div}(g) = Y + Y'$ for some Y' coprime with Y.

Corollary 2.4. *Let $Y = \sum m_iY_i$ be a divisor in X where $\{m_i\}$ is any sequence of integers. Then, there is $f \in \mathcal{M}(X)$ such that $m_{Y_i}(f) = m_i$ for all $i \in I$.*

Proof: Write $m_i = 2n_i$ or $m_i = 2n_i + 1$ according to the parity of m_i. By the theorem above, there is a sum of squares $h_- \in \mathcal{O}(X)$ such that $m_{Y_i}(h_-) = 2|n_i|$ for all $i \in I$ such that $n_i < 0$ with $\mathcal{Z}(h_-) = \bigcup_{n_i<0} Y_i$. Similarly there is a sum of squares $h_+ \in \mathcal{O}(X)$ such that $m_{Y_i}(h_+) = 2n_i$ for all $i \in I$ such that $n_i > 0$. Take $g \in \mathcal{O}(X)$ such that $m_{Y_i}(g) = 1$ when m_i is odd and not vanishing on any Y_i such that m_i is even.
Then, $f = h_+g/h_-$ has the required multiplicities. To check this just note that $m_{Y_i}(f) = m_{Y_i}(h_+) + m_{Y_i}(g) - m_{Y_i}(h_-)$. □

A similar result has been proved in [ADR] in the case of a real normal analytic surface X.

3 Locally principal divisors

In this section we assume that X is a coherent irreducible real analytic set in $\mathbb{R}^n$ of (pure) dimension q.
Let $Y \subset X$ be a reduced divisor; we are interested in the following question. Under what hypothesis is the ideal $I(Y)$ a principal ideal in $\mathcal{O}(X)$, that is, there is $g \in \mathcal{O}(X)$ such that $Y = \operatorname{div}(g)$? Note that if a global function g generates $I(Y)$ then, $X \setminus Y$ has at least two connected components and moreover the set $\{y \in Y | \dim_y Y = q-1\}$ bounds one of the regions where g has a given sign. So, in order to have $Y = \operatorname{div}(g)$ we must have that Y disconnects X and the class $[Y]$ vanishes in the group $\mathrm{H}^{\infty}_{q-1}(X, \mathbb{Z}_2)$. However these conditions are not sufficient as the following example shows.

Example 3.1. *Consider the set*

$$X = \{(x, y, z) \in \mathbb{R}^3 | z^2 = x^4 + y^4\}$$

and put $Y = \{p \in X | x = 0, z \geq 0\}$.
Y *is a parabola and* $[Y] = 0$ *in* $H^{\infty}_{q-1}(X, \mathbb{Z}^2)$. *This is clear since* Y *is the boundary of the open set* $\{x > 0, z > 0\} \cap X$. *Nevertheless the ideal* $I(Y)\mathcal{O}_{X,0}$ *is not principal; to see this one can apply [Mum], Prop. 2 pg. 384, or to check it directly. If it were principal, then,* $X \setminus Y$ *would split into two* principal *open semianalytic sets, which is not the case, as proved in [Per01].*

The previous example shows that another necessary condition for Y be principal is that the ideal sheaf $I(Y)\mathcal{O}_X$ is *locally principal*, that is, for any point $x \in X$ there is an function germ $f \in \mathcal{I}_{Y,x}$ that generates the stalk $\mathcal{I}_{Y,y}$ for any y in a neighbourhood of x. So, from now on we will assume this condition. In the case that X is not singular this condition follows from $[Y] = 0$ as it has been proved in [AB94]. In fact, there it is shown that in the nonsingular situation the condition $[Y] = 0$ already implies that $I(Y)\mathcal{O}_X$ is principal.
In the singular case we shall use some classical exact sequences of coherent sheaves and the vanishing of the cohomology of a coherent space.

Remark 3.2 It is easy to check that if $\{Y_i\}_{i \in I}$ is a locally finite family of locally principal divisors then $\bigcup Y_i$ is also a locally principal divisor. On the contrary, we can have a locally principal divisor with some components which are not locally principal.
For example, take $X \subset \mathbb{R}^3$ as the cone of equation $z^2 = x^2 + y^2$ and consider the divisor $Y = \{x = 0\} \cap X$ which is locally principal with generator $g = x$. The divisor Y splits into two straight lines Y_1 and Y_2 neither of which are locally principal.

We are ready to prove our main result.

Theorem 3.3. *Let X be a global coherent analytic set in $\mathbb{R}^n$ and $q = \dim X$. Assume that the ideal $I(X) \subset \mathcal{O}(\mathbb{R}^n)$ is prime. Let $Y \subset X$ be a reduced divisor such that its ideal sheaf $\mathcal{I}_Y$ is locally principal; assume that $[Y] = 0$ in $\mathrm{H}^{\infty}_{q-1}(X, \mathbb{Z}_2)$ and that $X \setminus Y$ is not connected. Then, there is $g \in \mathcal{O}(\mathbb{R}^n)$ such that $\mathcal{I}_Y = g\mathcal{O}_X$, in particular $Y = \mathrm{div}(g)$.*

Proof: Since $\mathcal{I}_Y$ is locally principal for any x there is a germ $f_x \in \mathcal{O}_{X,x}$ that generates $\mathcal{I}_{Y,y}$ for any y in an open neighbourhood of x. So, refining this one, we can find an open countable covering $\{U_i\}$ of X, analytic functions f_i on U_i such that $f_i\mathcal{O}_{X,x} = \mathcal{I}_{Y,x}$ for any $x \in U_i$; hence, f_i/f_j is invertible on $U_i \cap U_j$. These functions define an analytic cocycle in $H^1(X, \mathcal{O}^*)$, that is, an analytic line bundle $\mathcal{F}$ on X. We have to prove that $\mathcal{F}$ is trivial.
Consider the exponential map and the associated usual exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}^*_X \to \mathcal{O}^*_X/\mathcal{O}^+_X = \mathbb{Z}_2 \to 0,$$

Since $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, it induces an isomorphism between $H^1(X, \mathbb{Z}_2)$ and $H^1(X, \mathcal{O}^*_X)$. Under this isomorphism the image of a line bundle is the cocycle of the signs of its transition functions, so, to prove that $\mathcal{F}$ is trivial it is enough to find local generators $\{f_i\}$ of $\mathcal{I}_Y$ such that $f_i|_{U_i \cap U_j}$ and $f_j|_{U_i \cap U_j}$ have the same sign.
Take a locally finite triangulation $f : K \to X$ of the couple (X, Y); here K is a simplicial complex, and there is a subcomplex K_Y, such that $f(K_Y) = Y$. In particular, for any j, we have isomorphisms $f_* : \mathrm{H}^{\infty}_j(K, \mathbb{Z}_2) \to \mathrm{H}^{\infty}_j(X, \mathbb{Z}_2)$.
The fact that $[Y] = 0$ means that the union of all $q-1$ simplexes in K_Y bounds some subcomplex H of K. The boundary of the region $f(H) \subset X$ is the set $Y_{\max}$ of points in Y where Y has dimension $q-1$.
So, we may choose local generators $g_j \in \mathcal{O}(U_j)$ in such a way that g_j generates $\mathcal{I}_Y$ on U_j and it is positive on $f(H) \cap U_j \setminus Y$, while if $f(H) \cap U_j = \emptyset$, we choose g_j such that $g_j \geq 0$ when U_j lies in the same connected component as some component of $f(H)$ and $g_j \leq 0$ otherwise.
Hence $g_i/g_j > 0$ on $U_i \cap U_j$ and $\mathcal{F}$ is trivial. This means that we can find analytic functions $\{\lambda_i\} \in \mathcal{O}^*(U_i)$, such that $g_i/g_j = \lambda_j/\lambda_i$. So, the sections $g_i\lambda_i : U_i \to \mathcal{O}_X$ verify $g_i\lambda_i|_{U_i \cap U_j} = g_j\lambda_j|_{U_i \cap U_j}$, that is, they define an analytic function g on X and by construction g_x generates $\mathcal{I}_{Y,x}$ for any $x \in X$.

□

Remark 3.4 Note that when X is a manifold, the condition $[Y] = 0$ implies that Y divides X in two or more connected components and it is the boundary of some of them. Nevertheless this is not true in general, not even in the case of a coherent singular space X: as an example one can consider X to be a real 2-dimensional torus with one meridian collapsed to a point. One can easily write an analytic function on $\mathbb{R}^3$ with such a zero set. Take as Y any other meridian. Then, $[Y] = 0$, since it is homotopic to one point, and of course Y is locally principal, but it cannot be principal because its complement is connected.

As a consequence of Theorem 3.3 we have the following analogous of a result by Shiota ([Shi81]) that may be found in [BCR87], 12.4.1.

Corollary 3.5. *Let X be a global coherent analytic set in $\mathbb{R}^n$ and assume that the ideal $I(X) \subset \mathcal{O}(\mathbb{R}^n)$ is prime. Let $\mathfrak{p}_i, i \in \mathbb{N}$ be prime real ideals in $\mathcal{O}(X)$ of height 1. Denote by Y_i the associated divisor, i.e. the zero set of $\mathfrak{p}_i$, and assume that the family $\{Y_i\}_i$ is locally finite and that for any i the ideal sheaf $\mathfrak{p}_i\mathcal{O}_X$ is locally principal. Then, the ideal*

$$\prod_i \mathfrak{p}_i^{a_i}$$

is principal if and only if the cycle

$$\sum_i a_i[Y_i] = 0 \quad in \quad \mathrm{H}^\infty_{q-1}(X, \mathbb{Z}_2)$$

and

$$X \setminus \bigcup_{a_i \, odd} Y_i$$

is not connected.

Proof: Since the family of irreducible divisors $\{Y_i\}_i$ is locally finite, then, the ideal sheaf $\prod_i \mathfrak{p}_i^{a_i}\mathcal{O}_X$ is also locally principal. Put $a_i = 2k_i$ or $a_i = 2k_i + 1$ according to the parity of a_i. Split the class $\sum_i a_i[Y_i]$ as

$$\sum_i 2k_i[Y_i] + \sum_{a_i=2k_i+1} [Y_i].$$

Note that the ideal sheaf $\mathcal{J} = \prod_i \mathfrak{p}_i^{2k_i}\mathcal{O}_X$ is principal. In fact it is locally generated by a square, hence, arguing as in the proof of Theorem 3.3, its associated line bundle is trivial, which, in turn implies that we can find a global section g of $\mathcal{J}$ such that g_x generates its stalk at any point $x \in X$. Also, $Y = \sum_{a_i=2k_i+1} Y_i$. satisfies the hypothesis of Theorem 3.3, so its ideal is principal, say generated by f. So, fg generates $\prod_i \mathfrak{p}_i^{a_i}$ as wanted.
The converse is clear. □

Remark 3.6 Assume X is coherent and Y is locally principal, but not necessarily $[Y] = 0$. In this case we can find g such that $\mathrm{div}(g) = Y + Y'$ with Y' coprime with Y and reduced also. Indeed, as in the proof of Theorem 3.3, Y still defines a line bundle $\mathcal{F}$ on X. Now, using the general theory of bundles (see [Ste74]), isomorphism classes of line bundles are in one to one correspondence with homotopy classes of maps from the base space to the base space (the classifying space) of the corresponding universal bundle. Now, the universal bundle carries a natural structure of an analytic bundle over an analytic manifold. Since $\mathcal{F}$ is analytic it is classified by an analytic map from X to the classifying manifold; since X is coherent the classifying map extends analytically to an open neighborhood U of X in $\mathbb{R}^n$. This means that there is an analytic line bundle $\mathcal{E}$ on U such that $\mathcal{E}|_X$ is analytically isomorphic to $\mathcal{F}$. (cf. [Gua01]). Moreover the section of $\mathcal{F}$ defined by the local generators $\{f_i\}$ of $\mathfrak{I}_Y$ extends continuosly to a section of $\mathcal{E}$ ([Ste74] 12.2). Now, arguing as in [BP], we may find an analytic section $\{g_i\}$ of $\mathcal{E}$, transversal to the zero section and to X, whose zero set is an analytic manifold $W \subset U$. W cuts X along a reduced divisor Y' and it is easy to prove that the line bundle associated to $Y + Y'$ is trivial. So, the ideal of $Y \cup Y'$ is principal and its generator h verifies $\operatorname{div}(h) = Y + Y'$.

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